On derived graphs and digraphs

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The <u>derived</u> graph ∂G of a graph G has the edges of G as its vertices, with two vertices being adjacent if the corresponding edges are adjacent in G. Figure 1 shows a graph and its derivative. The derivative ∂D of a directed graph (or digraph) D has a similar

D e f i n i t i o n : The vertices are the arcs of D, with an arc from vertex a to vertex b in ∂D if in D the terminal vertex of arc a is the initial vertex of arc b.

Figure 2 illustrates the derivative of a digraph. Our aims are to present characterisations of derivatives in terms of subgraphs and to study classifications of iterated derivatives.

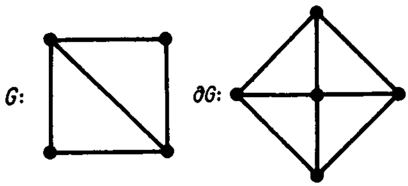

Fig. 1

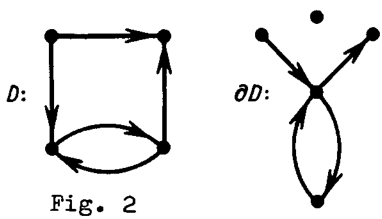

Fig. 2

I. Characterisations

The earliest characterisation of graph derivatives is due to KRAUSZ [1].

<u>Theorem 1:</u> A graph is a derivative if and only if its edges can be partitioned into complete subgraphs in such a way that no vertex lies in more than two of these subgraphs.

The second characterisation, due to van ROOIJ and WILF [2], gets more to the heart of the matter. One simple definition is required for its statement:

> Definition: A triangle of a graph is odd if some vertex of the graph is adjacent to an odd number of its three vertices.

Theorem 2: A graph is a derivative if and only if it does not have the star $K_{1,3}$ as an induced subgraph and whenever $a\ b\ c$ and $a\ b\ d$ are distinct odd triangles, then c and d are adjacent.

Our characterisation is actually a corollary of Theorem 2; its proof will appear elsewhere [3].

Theorem 3: A graph is a derivative if and only if none of the nine graphs of Figure 3 is an induced subgraph.

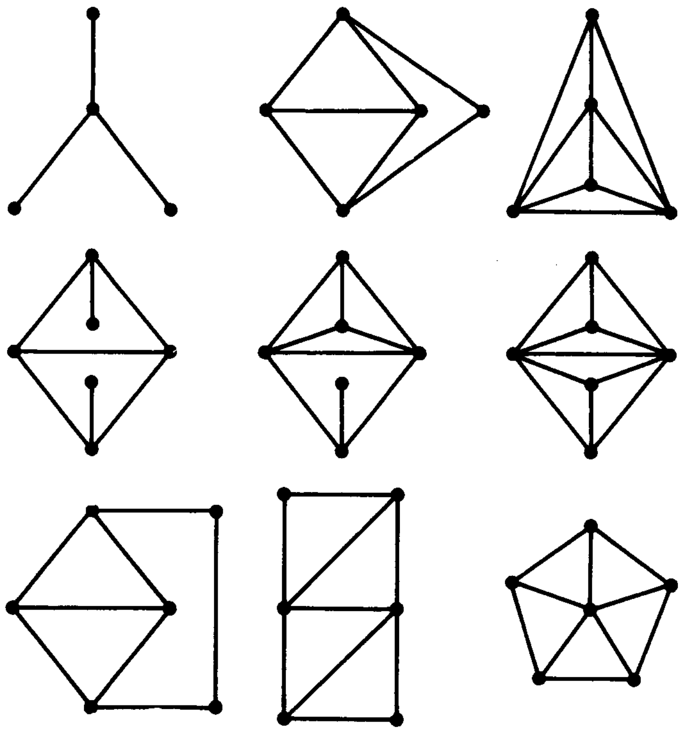

Fig. 3

We note that the first of these is $K_{1,3}$; all of the others have two odd triangles with a common edge but their other two vertices non-adjacent. Therefore none of the nine can be an induced subgraph. One characterisation of derived digraphs is analogous to Theorem 1, and due to HARARY and NORMAN [5]. If A and B are disjoint sets of vertices, K(A,B) denotes the digraph with vertex set $A \cup B$, with each vertex of A joined by arcs to all vertices of B.

Theorem 4: A digraph is a derivative if and only if its arcs can be partitioned into digraphs $K(A_i, B_i)$ in such a way that for all $j \neq k$, $A_j \cap A_k = \emptyset$. $B_j \cap B_k = \emptyset$, and $|A_j \cap B_k| \leq 1$.

HEUCHENNE's result [6] gives more insight into directed derivatives. His statement applies to those in which loops and multiple arcs are allowed, which we call general digraphs.

Theorem 5: A digraph is the derivative of a general digraph if and only if whenever a c, b c, and b d are arcs, so is a d.

As corollaries to this result, we give characterisation for the derivatives of less general classes. These can be stated in terms of subgraphs. An oriented graph is a digraph with at most one arc joining any two vertices, in other words, an irreflexive asymmetric relation.

Theorem 6: An oriented graph is a derivative if and only if none of the graphs of Figure 4 is an induced subgraph.

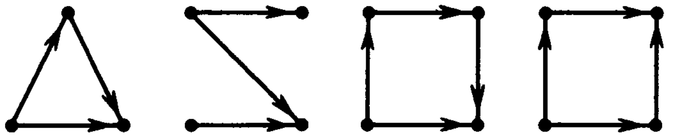

Fig. 4

The first three are excluded by HEUCHENNE's result, the fourth would necessitate multiple arcs. Next we consider digraphs in the ordinary strict sense.

Theorem 7: A digraph is a derivative if and only if none of the graphs of Figure 5 is a subgraph, and the first of the graphs of Figure 6 implies the second.

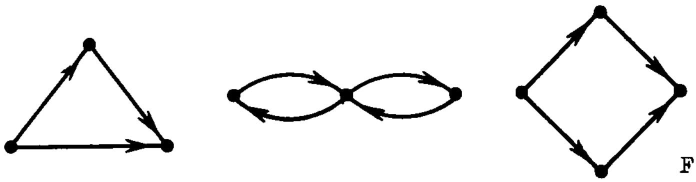

Fig. 5

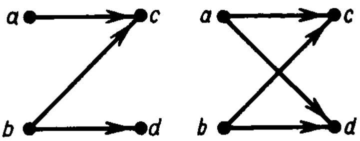

Fig. 6

II. Iterated Derivatives

In response to a question of ORE [4, p. 21] the results for iterated derivatives of graphs have been rediscovered many times. As an example, we take the classification of VAN ROOIJ and WILF [2]. The number of vertices in a graph G is denoted $|G|$.

Theorem 8: Let G be a connected graph.

(i) $|\partial^n G| \to 0$ if and only if G is a path.

(ii) $|\partial^n G| \to \infty$ if and only if the maximum valency is at least 3 and G is not $K_{1,3}$.

(iii) $G \cong \partial G$ if and only if G is a polygon.

(iv) $G \cong K_{1,3}$, in which case each iterate is a triangle.

The analogous classification can be made for digraphs, but the collection of those whose iterates neither vanish nor become arbitrarily large is of more interest. Such digraphs have periodic derivatives since there are integers m and n such that $\partial^m D \cong \partial^n D$. The smallest positive value $|n - m|$ is called the period of the derivatives.

Theorem 9: Let D be a digraph.

(i) $|\partial^n D| \to 0$ if and only if D has no cycles.

(ii) $|\partial^n D| \to \infty$ if and only if D has two cycles joined by a path.

(iii) D has periodic derivatives if and only if it has at least one cycle, but no two are joined by a path.

The proof of this result is straightforward and will not be given here. The same applies to many of the statements in the following discussion. (We note however that one must be careful. There may be some discrepancies between this and other literature, because certain existing "theorems" on this material are incorrect.)
We recall some facts about functional digraphs and arborescences [7]. An arborescence is a tree oriented so that one vertex can be reached by all others; see Figure 7. Its directional dual is called a counterarborescence. A functional digraph is one in which each vertex has outdegree 1. Equivalently, each weak component is unicyclic and consists of the cycle together with arborescences at each vertex of the cycle; see Figure 8.

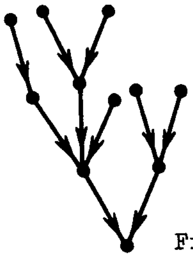

Fig. 7

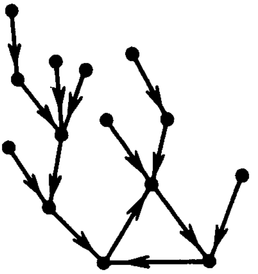

Fig. 8

HARARY and NORMAN [5] (and others [6], [8]) showed that all functional digraphs and their duals (contrafunctional digraphs) are isomorphic to their derivatives. While every such weakly connected digraph isomorphic to its derivative is functional or contrafunctional, there are many others which are disconnected. They all fit in the broader category of digraphs having derivatives of period 1. Some of these are obtained by the union of the k digraphs in a period; that is, if $\partial^m D \cong \partial^{m+k} D$, take the digraph

consisting of $\partial^m D$, $\partial^{m+1}D, \ldots, \partial^{m+k-1}D$. However there are some which are unicyclic, but neither functional nor contrafunctional, as for example, that in Figure 9. We want to determine those unicyclic digraphs having period 1.
Assume that D has periodic derivatives. Then the number of cycles and their lengths is the same in ∂D as in D. Furthermore, the number of vertices which can reach, or are reachable from, a

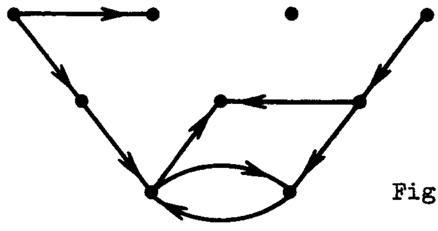

Fig. 9

particular cycle remains the same. Taking repeated derivatives, however, eventually results in digraphs in which no weakly connected component has more than one cycle. Therefore, we assume that D is unicyclic; and take E to be the subgraph consisting of all paths meeting the cycle. Eventually, $\partial^n D \cong \partial^n E$, so that we are primarily concerned with the structure of E and what information this gives.

In E every vertex is joined by a path to or from the cycle. There is thus associated with each vertex of the cycle the arborescence, consisting of all paths meeting the cycle first at that vertex, and the counter-arborescence of all paths from the vertex. Note that the arborescences need not all be disjoint, but they have only the vertices of the cycle in common with the counter-arborescences. These oriented trees give a criterion for unicyclic digraphs of period 1.

Theorem 10: A unicyclic digraph has period 1 if and only if all the arborescences or all the counter-arborescences are isomorphic.

Of course, the period of any unicyclic digraph can be given in these terms. The period is the least positive integer k such that the arborescences, or the counter-arborescences, at all vertices at distance k on the cycle are isomorphic.

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