On some metric problems in graph theory

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All considered graphs are non-oriented; we shall suppose no further restrictions for them. To every graph G with the vertex set $V(G)$ is assigned a triple (g_G, d_G, k_G) of cardinal numbers thus:

g_G (the girth of G) is the length of the shortest circuit in G; if G has no circuits - i.e. if G is a forest - we set $g_G = \aleph_0$.

d_G (the degree or the upper valency of G) is defined as $\sup_{v \in V(G)} d(v)$, where $d(v)$ denotes the degree of a vertex v. The symbol sup (the least upper bound) is related here to the class of all cardinal numbers.

k_G (the diameter of G) $= \sup_{u,v \in V(G)} \rho(u,v)$, where $\rho(u,v)$ is the length of the shortest path between the vertices u and v; if u and v are not connected by any path, we set $\rho(u,v) = \aleph_0$.

Let G be a graph. It is easy to prove that the following conditions are equivalent:

(1) Any two vertices of G are connected by at most one path *) of length $\leq k_G$;
(2) G has no circuits of length $\leq 2k_G$;
(3) the girth g_G of G is either $2k_G + 1$ or $\aleph_0$.

D e f i n i t i o n 1: A graph G fulfilling one of the equivalent conditions (1), (2) and (3) will be called a strongly geodetic graph. (For geodetic graphs, defined by means of the uniqueness of the shortest paths, see [2].)

*) In the sense of Ore [1].

Our aim is to characterize the class of all strongly geodetic graphs. The first result in this direction is

<u>Theorem 1:</u> If G is a strongly geodetic graph, then G is either a forest, or a regular graph with a finite diameter.

In the following we show that finite strongly geodetic graphs are closely related to the so-called Moore graphs, whose study was suggested by E.F. MOORE.

Let finite cardinal numbers d and k be given. It can be easily found that for the number n of vertices of any graph of degree d and with diameter k we have

$$n \leq 1 + d \sum_{i=1}^{k} (d-1)^{i-1} \qquad (*)$$

For some pairs (d, k) there exist graphs for which we have in (*) an equality; these graphs are called Moore graphs of type (d, k). For example, the Petersen graph is a Moore graph of type $(3, 2)$.

Moore graphs with diameters 2 and 3 - with the exception of the only type $(57, 2)$ - were described in [3] by HOFFMAN and SINGLETON. They also constructed a regular graph of degree 7 with diameter 2 and with 50 vertices, i.e. a Moore graph of type $(7, 2)$ and proved its uniqueness; we shall therefore call this graph the Hoffman-Singleton graph.

For the present we are able to add only a small contribution to the theory of Moore graphs, namely

<u>Theorem 2:</u> For $3 < k < 8$ no Moore graph of type $(3, k)$ exists.

The relation between strongly geodetic and Moore graphs is described in

<u>Theorem 3:</u> A graph G is a Moore graph if and only if G is a finite, connected, regular and strongly geodetic graph.

In other words Moore graphs may be characterized as finite regular strongly geodetic graphs with a finite diameter. Let us form an analogical class in the case of infinite graphs.

D e f i n i t i o n 2: By a quasi-Moore graph we mean an infinite regular strongly geodetic graph with a finite diameter.

Obviously all quasi-Moore graphs can be classified by means of their degree and diameter into types (d, k), where $d \geq \aleph_0$, $k < \aleph_0$.

Theorem 4: For any cardinal numbers $d \geq \aleph_0$, $k < \aleph_0$ there exists a quasi-Moore graph of type (d, k).

In the proof of this theorem a construction is used that reminds of the process of building a spider's web.
From Theorem 1, Theorem 3 and Definition 2 follows:

Theorem 5: A graph G is strongly geodetic if and only if G is either a forest, or a Moore graph, or a quasi-Moore graph.

From Theorem 2, Theorem 5 and the results of [3] we obtain

Corollary: A graph G is strongly geodetic if and only if one of the following 10 cases takes place:

1^0 G is a forest;
2^0 G is a complete graph;
3^0 G is an odd polygon (i.e. G is formed by vertices and edges of a circuit with an odd number of its edges);
4^0 G is formed by the only vertex and by loops;
5^0 G is isomorphic to the Petersen graph;
6^0 G is isomorphic to the Hoffman-Singleton graph;
7^0 G is a quasi-Moore graph;
8^0 G is a Moore graph of type $(57, 2)$;
9^0 G is a Moore graph of type $(3, k)$, where $k > 7$;
10^0 G is a Moore graph of type (d, k), where $d > 3$, $k > 3$.

The existence of graphs of the cases $8^0 - 10^0$ remains an unsolved problem. The second unsolved problem is whether the graphs of $7^0 - 10^0$ (if they exist) are uniquely determined (up to isomorphism) by their degree and diameter.

R e m a r k: The proofs will be published in [4].

References

[1] Ore, O.: Theory of graphs, Amer. Math. Soc., Providence 1962.

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[3] Hoffman, A.J.; Singleton, R.R.: On Moore graphs with diameters 2 and 3, IBM Journal of research and development 4 (1960), Nr. 5, 497-504.

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