On the completeness – number of a finite graph

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The aim of this contribution is to solve (partially) the problem Nr. 3 from Smolenice ([1], p. 157).

The graphs considered are finite, non-directed, without loops and multiple edges. The completeness-number $\omega(G)$ of the graph G is defined as follows:

D e f i n i t i o n 1: The system $\mathfrak{G}$ of the complete subgraphs of the graph $G = \langle V, E \rangle$ covers G, if every vertex $v \in V$ and every edge $e \in E$ belongs to some subgraph $F \in \mathfrak{G}$. The smallest cardinality of the system $\mathfrak{G}$ covering G is called the completeness-number of the graph G and denoted by $\omega(G)$.

Dr. ČULIK, who formulated the problem, asked

1) to find a non-trivial algorithm for the determination of $\omega(G)$
2) to determine the relations between $\omega(G)$ and the other fundamental characteristics of graphs.

What follows, deals with the second half of the question. It is possible to construct to a given graph G a new graph G', such that $\omega(G) = \chi(G')$ where χ denotes a chromatic number. In order to carry out the construction an auxiliary notion is needed.

D e f i n i t i o n 2: Two edges e_1, e_2 of the graph G are quasineighbours, if $e_1 \neq e_2$ and e_1, e_2 both belong to a certain complete subgraph of G.

The two possibilities for edges e_1, e_2 to be quasineighbours are shown at the figure 1.

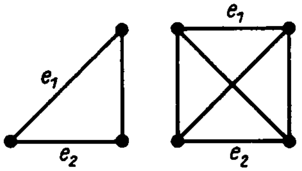

Fig. 1

Note: Let $G = \langle V,E \rangle$, $v \notin V$, $G_1 = \langle V \cup \{v\}, E \rangle$; let v be an isolated vertex of the graph G_1. Then obviously, $\omega(G_1) = \omega(G) + 1$.

On investigating the completeness-number of the graph G we are enabled, according to the note above, to pass over to the graph which is obtained by removing the isolated vertices of the graph G.

Let $G = \langle V,E \rangle$ be the graph without isolated vertices. We shall denote by $G' = \langle V',E' \rangle$ the graph, which satisfies the following conditions:

Condition 1: The edges of G correspond uniquely to the vertices of G' (let us denote the one-to-one mapping of the set E onto the set V' by φ).

Condition 2: The vertices v_1', v_2' of the graph G' are connected by a certain edge in G' if and only if the corresponding edges, i.e. $\varphi^{-1}(v_1')$ and $\varphi^{-1}(v_2')$ are not quasineighbours in G.

If G is an arbitrary graph without isolated vertices, then obviously there exists just one graph possessing the required properties (with the exception of isomorphism).

Now, the following theorem may be easily proved ([2]):

Theorem: Let G be the graph without isolated vertices. Then $\omega(G) = \chi(G')$, where G' is the graph satisfying the conditions 1 and 2 and $\chi(G')$ is its chromatic number.

N o t e : Let $G_1 = \langle V,E \rangle$ be an arbitrary graph. There need not always exist such a graph G that $G' = G_1$. The necessary conditions for the existence of such a graph is the following

C o n d i t i o n 3: The intersection of an arbitrary system of maximum (in the sense of inclusion) complete subgraphs of the graph G_1^* has a number of vertices equal to some of the numbers $0, 1, 3, \ldots, \frac{1}{2}k(k-1), \ldots$. Here G_1^* is the complementary graph of G_1 considered without loops, by the intersection of the system consisting of one maximum complete subgraph we understand the subgraph itself and the numbers $0, 1, 3, \ldots, \frac{1}{2}k(k-1), \ldots$ determine the number of edges in a complete subgraph of the order $1, 2, \ldots, k, \ldots$.
Obviously, the condition 3 is not sufficient.

Let us call the edges $e_1, \ldots, e_k \in E$ <u>independent</u>, if no two of them are quasineighbours. The question is, what is the maximum number of independent edges of a graph G without isolated vertices. One can easily show, that it is never greater than $\omega(G)$, since no two of independent edges can belong to the same complete subgraph. Actually, if $\omega(G) \le 4$, then the maximum number of independent edges is equal to $\omega(G)$. On the other hand, there is a graph G having at most 4 independent edges such that $\omega(G) = 5$ (cf. fig. 2).

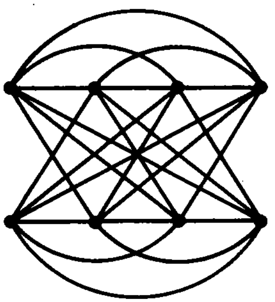

Fig. 2

References

[1] Fiedler, M. ed.: Theory of graphs and its applications. Proceedings of the Symposium held in Smolenice (Czechoslovakia), June 17-20, 1963. Nakladatelství ČSAV, Praha 1964.

[2] Havel, I.: On the completeness-number of a finite graph. Časopis pěst. mat. 90(1965), 191-193.