$K_{s,t}$-saturated bipartite graphs

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April 10, 2014

joint work with Wenying Gan and Benny Sudakov

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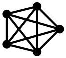

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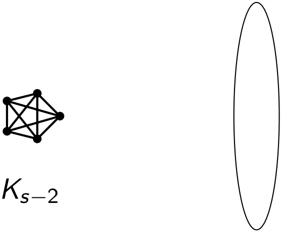

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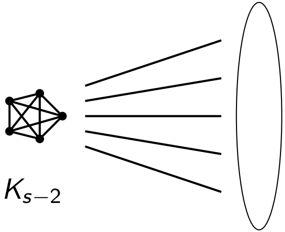

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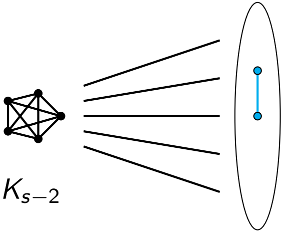

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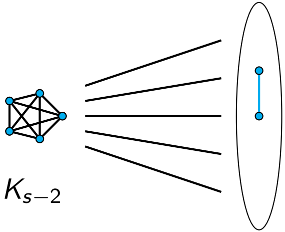

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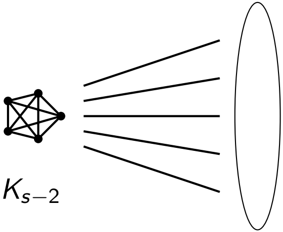

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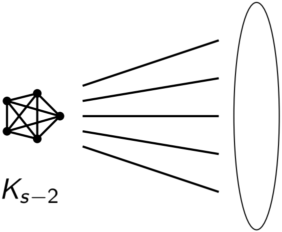

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$$= (s-2)n - O(1)$$

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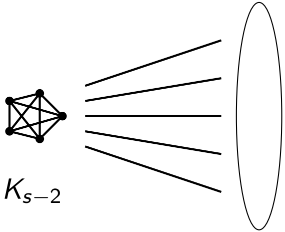

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Theorem (Zykov, 1949; Erdős–Hajnal–Moon, 1964)

A K_s-saturated n-vertex graph has at least $\binom{n}{2} - \binom{n-s+2}{2}$ edges.

Weak saturation

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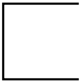

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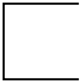

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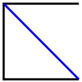

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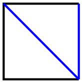

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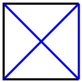

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Various proofs by Alon, Frankl, Kalai, Yu, etc.

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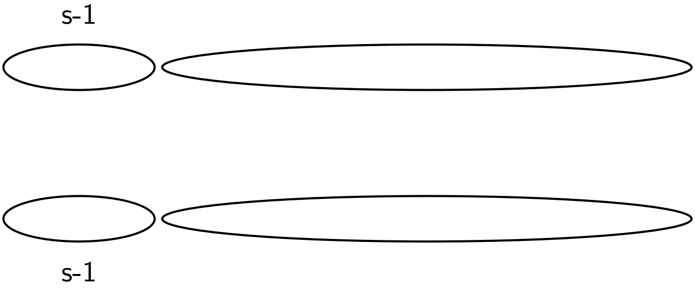

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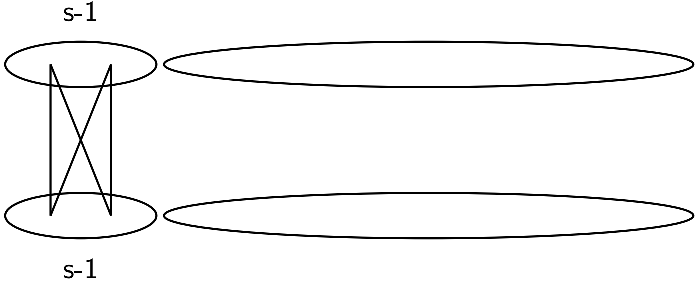

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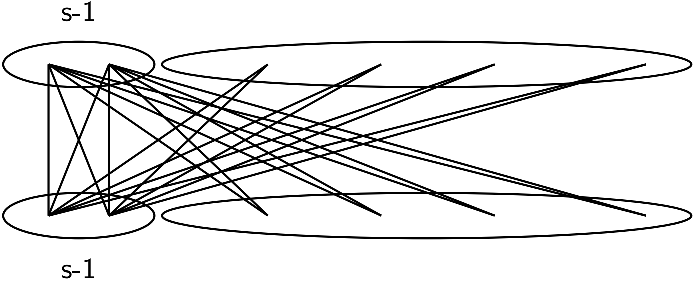

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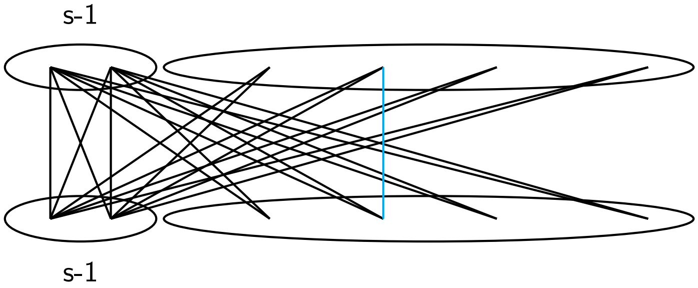

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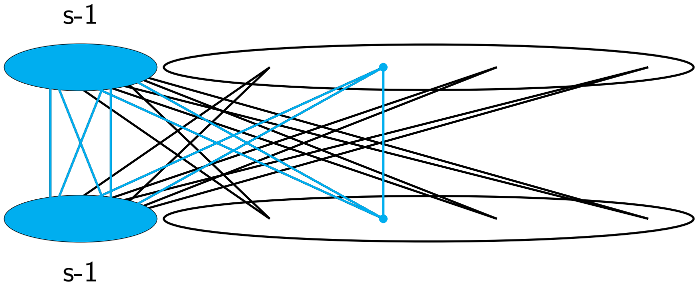

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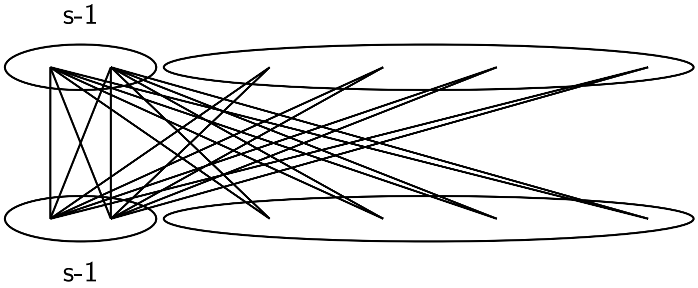

$n^2 - (n-s+1)^2 = (2s-2)n - O(1)$ edges.

Bipartite saturation

Theorem (Wessel, 1966; Bollobás, 1967; Alon, 1985)

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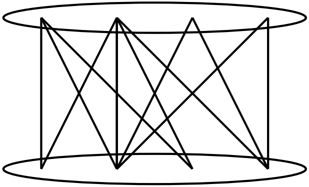

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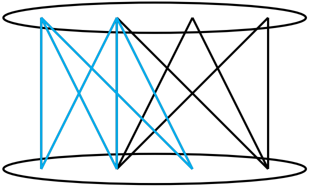

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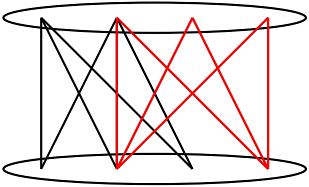

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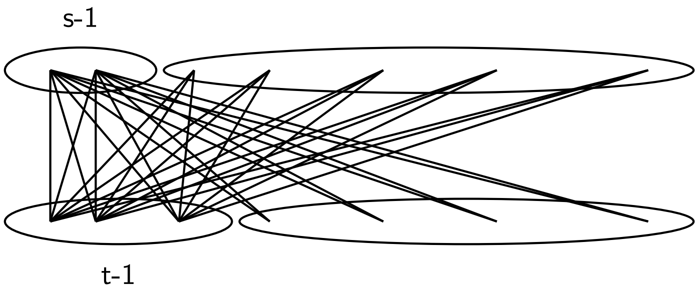

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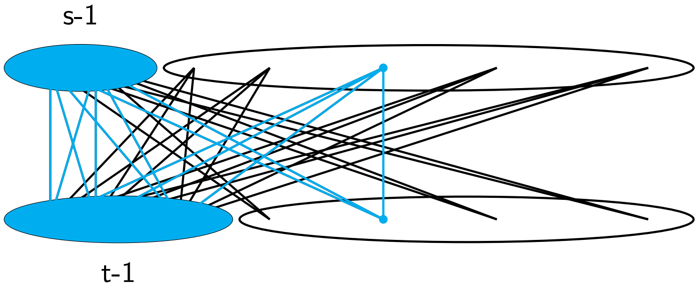

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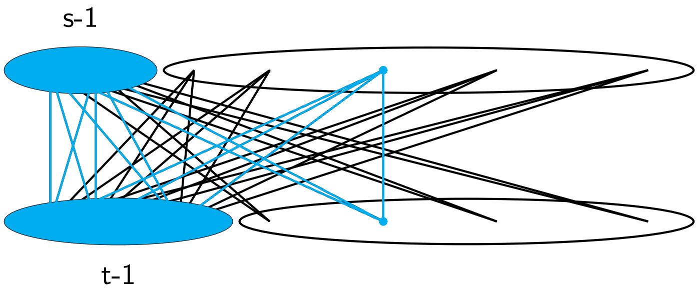

$(s+t-2)n - (s-1)(t-1)$ edges

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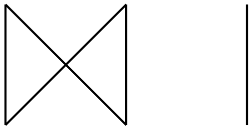

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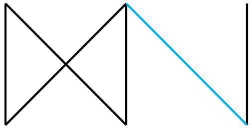

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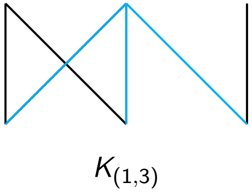

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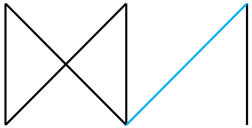

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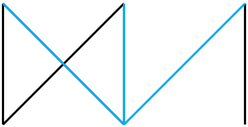

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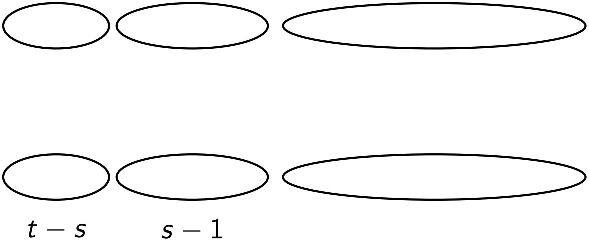

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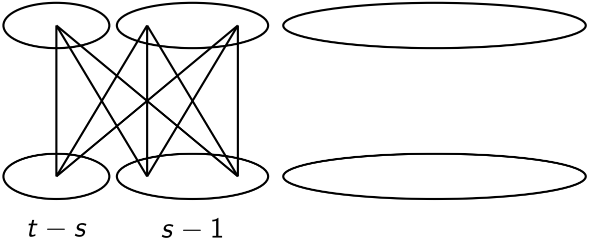

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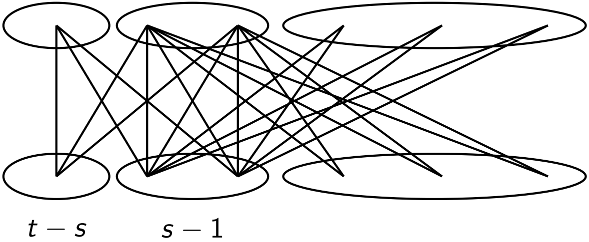

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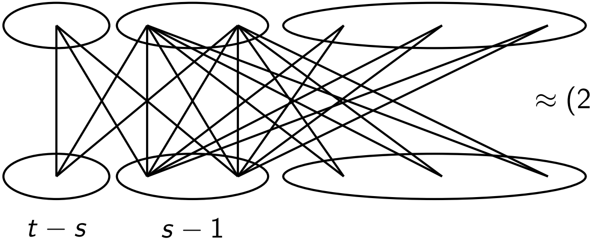

$\approx (2s-2)n$ edges

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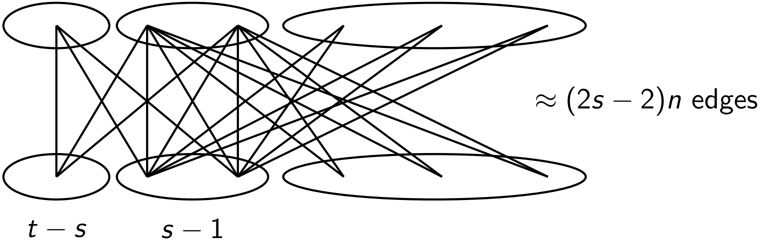

- ordered case: at least $(s+t-2)n - O(1)$ edges

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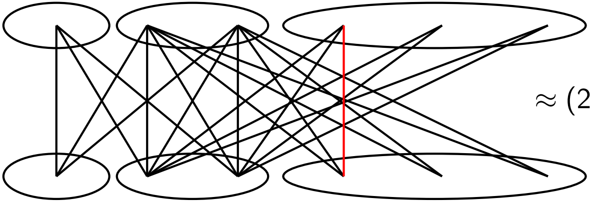

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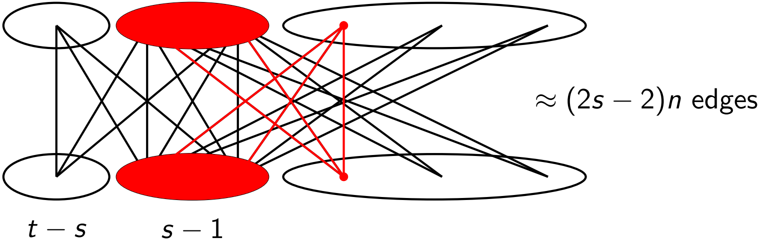

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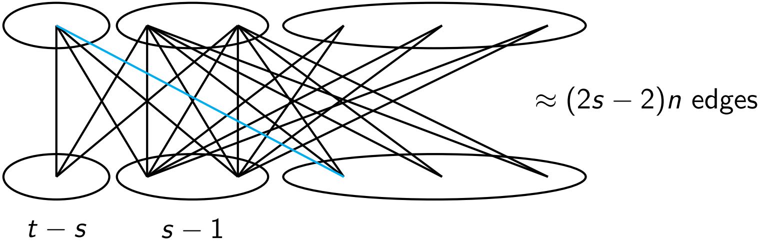

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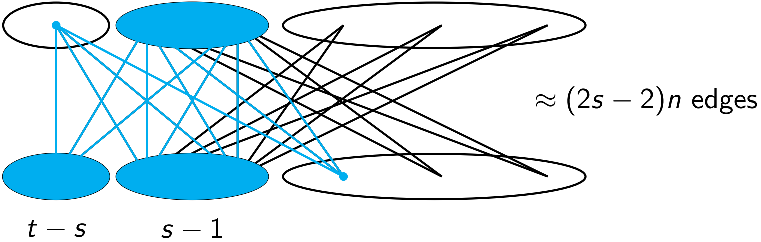

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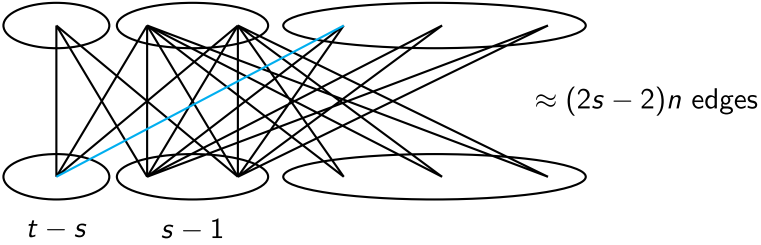

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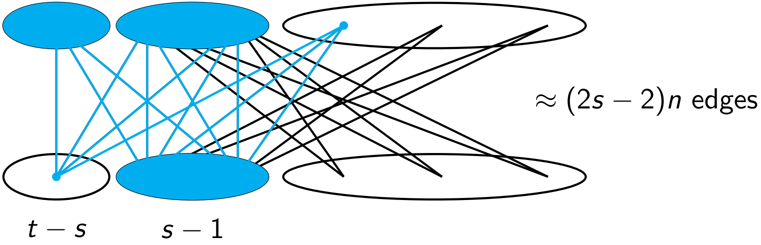

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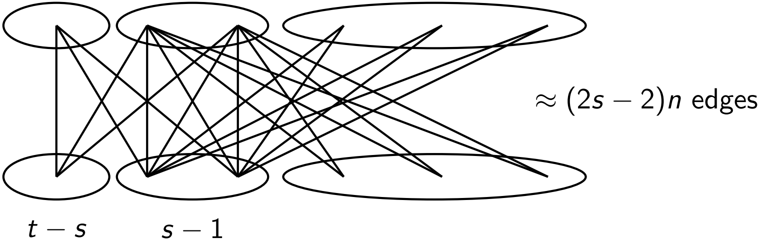

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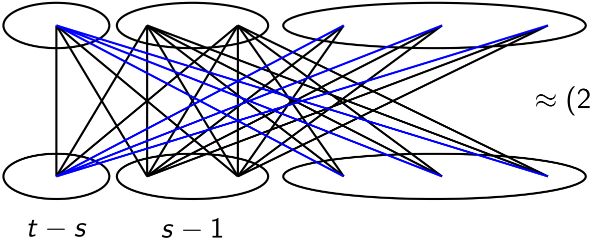

$\approx (2s-2)n$ edges

- ordered case: at least $(s+t-2)n - O(1)$ edges

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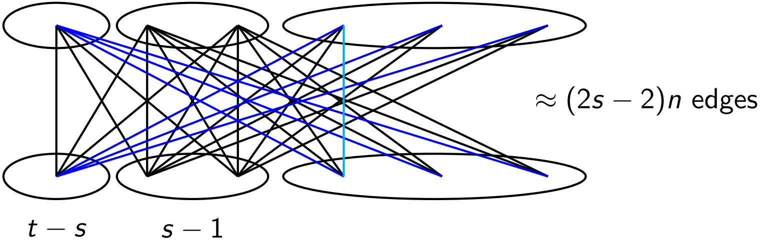

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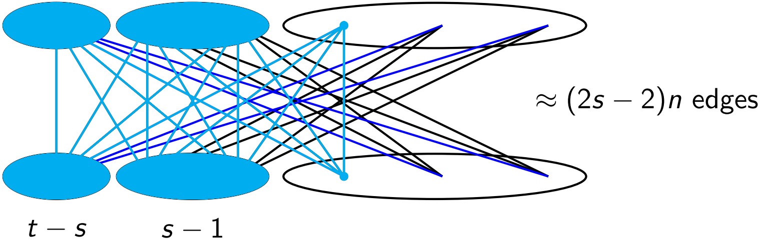

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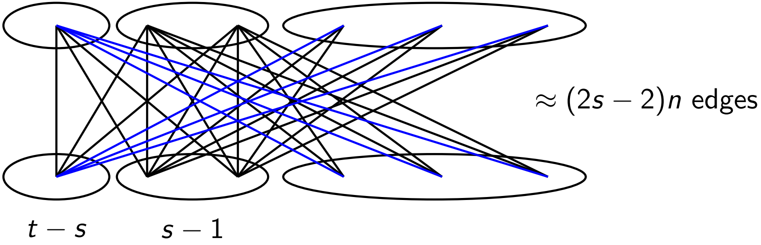

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Theorem (Moshkovitz–Shapira, 2014+)

A weakly $K_{s,t}$-saturated n-by-n bipartite graph has at least $n^2-(n-s+1)^2+(t-s)^2$ edges.

Strong $K_{s,t}$-saturation

Conjecture (Moshkovitz–Shapira)

A $K_{s,t}$-saturated n-by-n bipartite graph has at least $(s+t-2)n - \lfloor \left(\frac{s+t-2}{2}\right)^2 \rfloor$ edges.

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Theorem (Gan–K–Sudakov, 2014+)

A $K_{s,t}$-saturated n-by-n bipartite graph contains at least $(s+t-2)n - (s+t-2)^2$ edges, and this is tight up to an additive constant.

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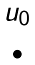

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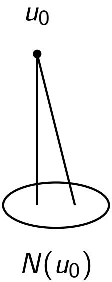

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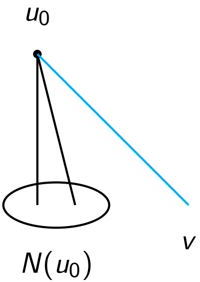

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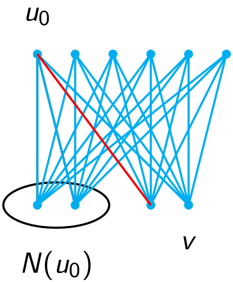

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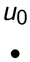

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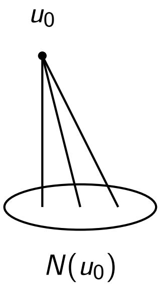

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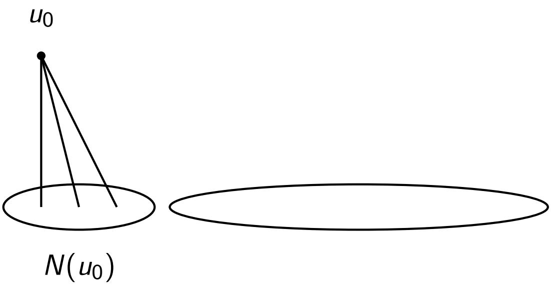

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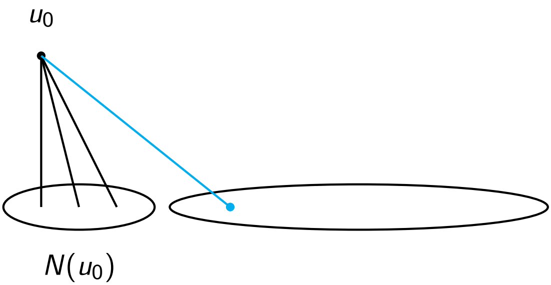

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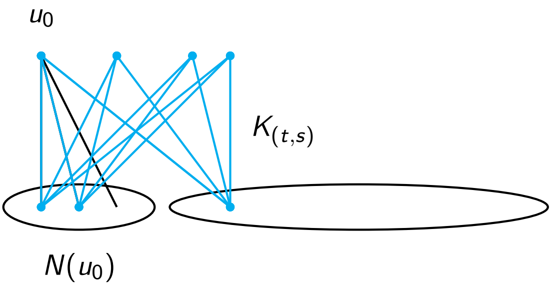

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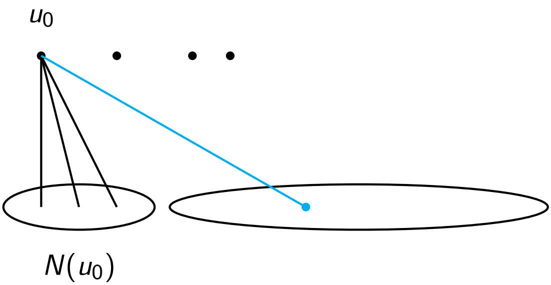

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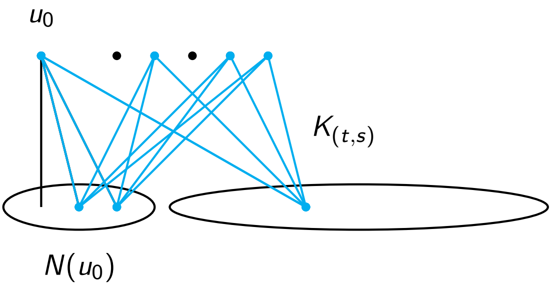

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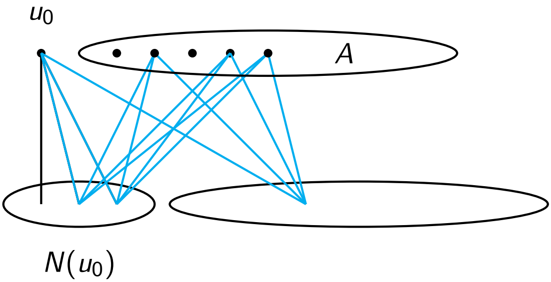

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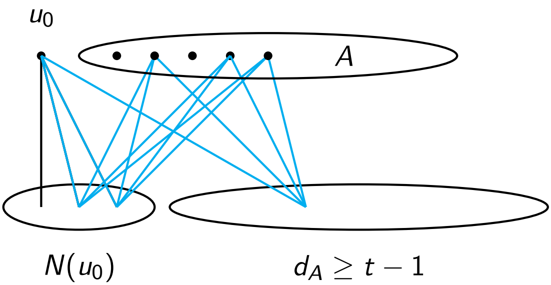

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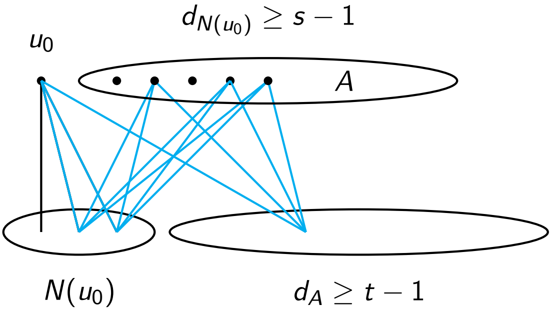

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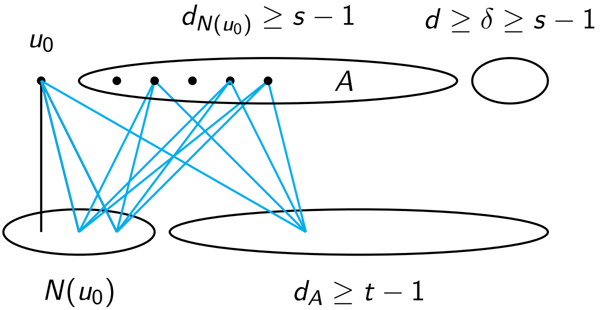

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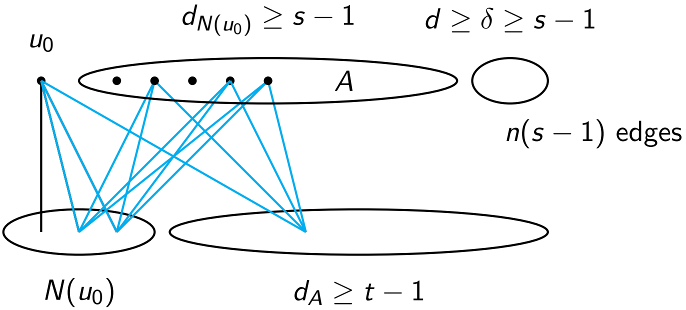

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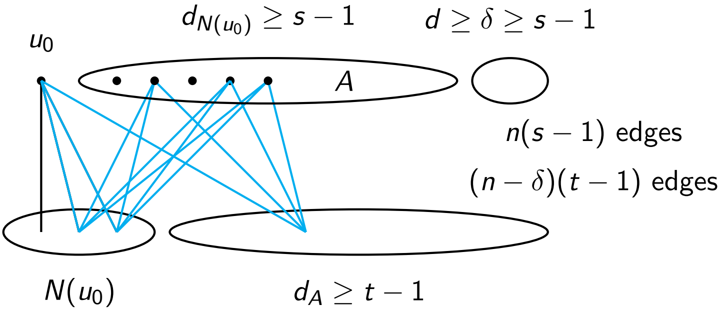

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- but first we need a symmetric (and more complicated) structure for counting the edges
- and then decide which class is which.

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2. Generalize to hypergraphs.

Thank you!