Weakly k - saturated graphs

Béla Bollobás

The term graph will be used throughout this paper for edge-graphs without loops and multiple edges. A graph is called saturated with respect to a property P if the graph does not have property P, but it will have this property when any new edge is added to the graph. A great number of the extremal problems in graph theory can be formulated as follows: How many edges at most (or at least) can (or must) a graph with n vertices have if it is saturated with respect to a certain property? One of the most natural properties is that the graph contains a complete k-graph. The graphs saturated with respect to this property are called k-saturated graphs. The celebrated theorem of TURÁN [1] establishes the maximum number of edges of a k-saturated graph. On the other hand, ERDŐS, HAJNAL and MOON [2] calculated the minimum number of edges a k-saturated graph must contain. This second question could be answered for generalized graphs [3], and by proving the conjecture of ERDŐS, HAJNAL, and MOON, for bipartite graphs, too (see [4], [5], [6], and [7]). There is also a very natural conjecture for r-partite generalized graphs (where r-tuples are considered instead of the edges), but the proof breaks down even for $r = 3$, because a different concept of saturatedness is needed. It was this which partly motivated the concept of weakly saturated graphs which we shall introduce in the following and about which we shall prove some initial results.

Consider a graph with n vertices and add all those edges which are the only missing edges of complete k-graphs (i.e. we add the edge α if there are k such vertices of the graph, that the graph contains all the edges spanned by these k vertices, saving α). If by repeating this process a sufficient number of times the complete n-graph is obtained, the original graph will be called weakly k-saturated. Denote by $f(n,k)$ the minimum number of edges of these graphs. It is obvious that every k-saturated graph is also weakly k-saturated since all the edges are obtained already after the first step. Before determining $f(n,k)$ for certain values of k, we give another interpretation of the problem.

Let us call a party <u>shy of order s</u> if two persons can be introduced to each other only by gathering s more members of the party, knowing each other and these two people, too. At least how many acquaintances are needed at the beginning of a party shy of order s, if the party consists of n people and we want that at the end everybody knows everybody else, provided all possible introductions are made? This number was denoted by $f(n,s+2)$.

<u>Theorem:</u> If $3 \le k < 7$, $k \le n$ then $f(n,k) = (k-2)n - \binom{k-1}{2}$.

Obviously, this theorem is a sharpening of the theorem of ERDŐS, HAJNAL and MOON.

P r o o f : Consider the following graph: take a complete $(k-2)$-graph and add $n - k + 2$ vertices, each of which is connected with the vertices of the complete graph (see Fig. 1).

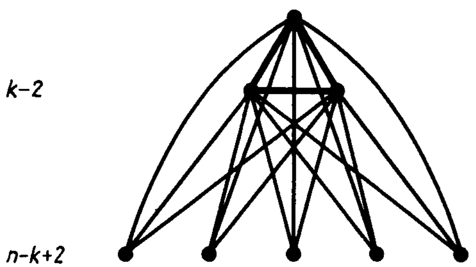

Fig. 1

This graph is k-saturated, consequently weakly k-saturated, and has $(k-2)n - \binom{k-1}{2}$ edges. Thus, $f(n,k) \ge (k-2)n - \binom{k-1}{2}$ for all $k \ge 3$.

The proof uses induction on n. If $n = k$, the theorem is trivial. If $n > k$, take a weakly k-saturated graph, G say, with minimum number of edges. It is sufficient to show that G has at least $k - 2$ more edges than a weakly k-saturated graph with $n - 1$ vertices.

Suppose that the vertex a has minimum degree, d say. As $\frac{n \cdot d}{2} \leq f(n,k) \leq (k-2)n - \binom{k-1}{2}$, $d \leq 2k - 5$. On the other hand, naturally $k - 2 \leq d$. Put $d = k - 2 + p$, then $0 \leq p \leq k - 3$.

The gist of the proof is the following: Add edges (which are the only missing edges of graphs spanned by k vertices) one by one and notice, when use is made of the vertex a. The use of a in $G - a$ can be substituted by adding a certain edge to $G - a$ at the very beginning. To prove the theorem it is sufficient to show that by adding edges at most p times to $G - a$, it is not necessary to use a any more to get a complete graph of $G - a$, i.e. this new $G - a$ graph is weakly k-saturated.

Let us call a complete graph of $G - a$ a <u>clique</u> if it is a maximal complete subgraph with at least $k - 2$ vertices.

The term <u>base</u> will be used for the set of vertices connected to a at the different stages of the procedure.

Adding edges to G means that if two complete subgraphs having at least $k - 1$ vertices meet in at least $k - 2$ vertices, then the whole subgraph spanned by these vertices can be completed.

The following statements are immediate consequences of this fact and the definitions above, and describe those steps of the procedure of filling G with edges which may be effected by the omission of the vertex a.

1. The base can be enlarged as follows: if a clique has at least $k - 2$ vertices in the base, then the new base will be the union of the clique and the base (Fig. 2).

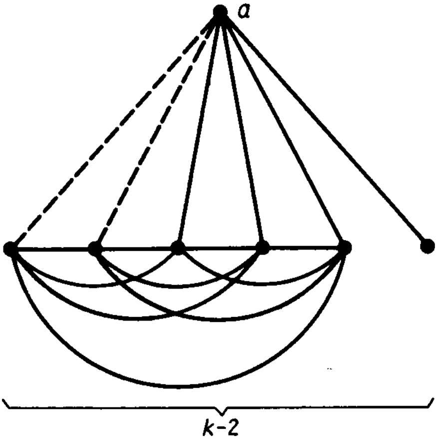

Fig. 2

2. If two cliques of the base have exactly $k - 3$ vertices in common, then by using the vertex a, these two cliques can be united. If we want to omit a, this union can be obtained by adding to the original graph $G - a$ an edge α, connecting such vertices of the two cliques which are not in the common part (Fig. 3).

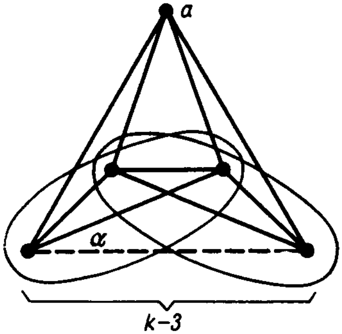

Fig. 3

If after having added certain edges to the graph, the whole base is a single clique, then $G - a$ can be completed without the use of a, since the application of step 1 gives again a single clique and this implies that step 2 is not used any more. Thus, in order to prove the theorem it would be sufficient to verify that after having used step 2 at most p times, the whole base will be a single clique. It is obvious that instead of adding edges to the graph, we may suppose that cliques are added, having at least $k - 2$ vertices in the base. This process must go on until the base contains $n - 1$ vertices and it is a single clique.

As $0 \leq p \leq k - 3$ and $k \leq 6$, $p = 0, 1, 2$ or 3. A clique will be called new if it is not the enlargement of a previous one (as in step 1) and they will be denoted by $C_1, C_2, \ldots$ in the order of their appearance.

a) If $p = 0$, then C_1 will contain all the vertices of the base.

b) If $p = 1$, the first two cliques, C_1 and C_2, have at least $k-3$ vertices in common, so they must be united according to step 2, and their union is the whole base.

c) If $p = 2$, then $k = 5$ or 6.

If the first two cliques, C_1 and C_2, have $k - 3$ common vertices then, by omitting an edge starting from a and adding a suitable edge to $G - \underline{a}$, a weakly k-saturated graph will be obtained, which has the same number of edges as G, and for which already $p = 1$. So it may be supposed that C_1 and C_2 have at most $k - 4$ vertices in common. Therefore they have exactly $k - 4$ vertices in common, thus covering the whole base.

If $k = 5$, i.e. if the cliques have at least 3 vertices, the third clique, C_3, must have at least two common vertices with one of C_1 and C_2. So they must be united according to step 2, and step 2 will be used again to unite it with the remaining one. But this makes the whole base a single clique.

If $k = 6$, the cliques have at least 4 vertices. In case C_3 has three vertices in common with one of the first cliques, then the same situation arises as above. Otherwise by suitably numbering the vertices it can be supposed that the cliques are the following: (1,2,3,4), (1,2,5,6), (3,4,5,6) and certain additional vertices to any of the cliques. In this case the graph contains all the edges of the complete graph spanned by the vertices 1,..., 6, and as this clique C has four vertices in common with each of the first three cliques, the whole base can be made a single clique, without using step 2 at all.

d) If $p = 3$, then $k = 6$, the initial base has 7 vertices, say 1,2,3,4,5,6,7.

Just as in paragraph c) it can be assumed that C_1 and C_2 have at most two vertices in common, and so they have one or two common vertices.

If C_1 and C_2 have two vertices in common, we may suppose that they are (1,2,3,4), (3,4,5,6) and some additional vertices, and the base is enlarged with exactly these additional vertices. Now, if C_3 has three vertices in common with one of the first two cliques, then we essentially get the case already excluded.

Otherwise the third clique must contain two vertices like 1 and 5. But then (1,3,4,5) is also a clique which can be united according to step 2 with the first two cliques. The clique thus obtained, C say, contains all the vertices of the base, saving at most vertex 7. Finally, C_4 must contain vertex 7 and must have at least three vertices in common with C. In consequence their union must be formed according to step 2, and this union will be the whole base.

If C_1 and C_2 have one common vertex, then C_3 must have at least two vertices in common with at least one of them. But then the order of the cliques and the base itself can be changed in such a way as to obtain a previously discussed case. For instance, if $C_1 = (1,2,3,4,8,9,10)$, $C_2 = (4,5,6,7,11,12)$, and $C_3 = (8,9,11,12)$, then take $(6,7,8,9,10,11,12)$ as the new base and $C_1' = C_3$, $C_2' = C_2$, $C_3' = C_1$. This can be done since after having added these cliques, the base will be the same as after the original start. This completes the proof of the theorem.

For $k = 4, 5, 6$ we could not determine the set of extremal graphs, but a fairly large number of extremal graphs can be given as follows. If $n = k$, the only extremal graph is the graph from which one edge is missing. If G is an extremal graph of $n - 1$ $(\geq k)$ vertices, then add a vertex to G and join this vertex to any $k - 2$ of the vertices in G. The graph obtained will be an extremal graph with n vertices (Fig. 4). We do not know whether these are the

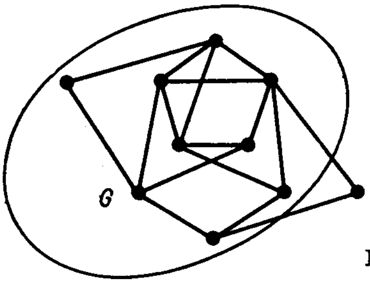

Fig. 4

only extremal graphs or not. If $k = 3$, it is obvious from the proof of the theorem that these are the only extremal graphs (the trees), since in this case $p = 0$.
Though the discussional method, which was applied in the proof, can not be carried out in the case $k \geq 7$, it seems to be likely that $f(n,k) = (k-2)n - \binom{k-1}{2}$ even for some further small values of k.

References

[1] Turán, P.: On an extremal problem in graph theory (in Hungarian), Mat. Fiz. Lapok, 48 (1941), 436-452.

[2] Erdős, P.; Hajnal, A.; Moon, J.W.: A problem in graph theory, Amer. Math. Monthly, 71 (1964), 1107-1110.

[3] Bollobás, B.: On generalized graphs, Acta Math. Acad. Sci. Hung. 16 (1965), 447-452.

[4] Bollobás, B.: On a conjecture of Erdős, Hajnal and Moon, Amer. Math. Monthly, 74 (1967), 178-179.

[5] Bollobás, B.: Determination of extremal graphs by using weights (to appear).

[6] Wessel, W.: Über eine Klasse paarer Graphen I: Beweis einer Vermutung von Erdős, Hajnal und Moon. Wiss. Z. TH Ilmenau 12 (1966), 253-256.

[7] Wessel, W.: Über eine Klasse paarer Graphen II: Bestimmung der Minimalgraphen. Wiss. Z. TH Ilmenau (im Druck).