The planar and toroidal crossing numbers of K_n

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This note is a summary of references [1,2] given below. The crossing number, cr(G), of a graph G has been defined [3] as the least number of crossings, or common points of arcs other than nodes, in any drawing, of G on a plane or sphere, in which the nodes of G are mapped into points, and the arcs into continuous curves on the plane. A minimal drawing is one in which the crossing number is attained; it does not contain an arc which crosses itself, nor two arcs with more than one point in common. These definitions extend to mappings on orientable surfaces of genus γ, in particular, $\gamma = 1$, to the toroidal crossing number, $cr_1(G)$. We also define the corresponding geodesic crossing numbers, $\overline{cr}(G)$, in which the arcs are mapped into geodesics of the surface.

It is known [1] that

$$cr(K_n) \leq \frac{1}{4} [\tfrac{1}{2} n][\tfrac{1}{2} (n-1)][\tfrac{1}{2} (n-2)][\tfrac{1}{2} (n-3)], \qquad (1)$$

where K_n is the complete graph on n nodes, and brackets mean 'entier', but the following construction, which also yields (1), is believed to be new. Consider the vertices of a regular n-gon. Join them by straight segments if these are in directions within a given quadrant, and otherwise make joins outside (e.g., the inverses of straight joins). It has been verified that equality holds in (1) for $n \leq 16$, and it can be shown that

$$cr(K_n) \geq \frac{11}{34} \binom{n}{4}, \qquad n \geq 17. \qquad (2)$$

However, KAINEN [4] has recently shown that

$$\lim_{n \to \infty} cr(K_n)/n^4 = \frac{1}{64} . \qquad (3)$$

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For the geodesic crossing number, HARARY and HILL [3] raised the question on the plane, while MOON [5] threw light on the sphere, showing

$$\overline{cr}(K_n) \le \frac{3}{8}\binom{n}{4}, \tag{4}$$

so that the limit (3) is the same for $\overline{cr}(K_n)$ on the sphere, whereas on the plane the situation may be different [3].

On the torus, actual drawings show that for

n	$=$	7	8	9	10	11	12	13	14	15	16
$cr_1(K_n)$	$\le$	0	4	9	23	42	70	105	154	226	326.

Equality can be established in the first four cases, e.g. suppose that K_9 can be drawn on the torus with m crossings, inducing a map with $V = 9 + m$ vertices, $E = \binom{9}{2} + 2m$ edges, and hence $F = 27 + m$ regions, so that $3(27 + m) = 3F \le 2E = 2(45 + 2m)$, and $m \ge 9$. Whether $cr_1(K_{11}) = 42$ remains a strong conjecture. From these results the lower bound

$$cr_1(K_n) \ge \frac{4}{35}\binom{n}{4}, \quad n \ge 15, \tag{5}$$

can be deduced, while actual construction shows

$$cr_1(K_n) \le \frac{59}{216}\binom{n-1}{4}. \tag{6}$$

The methods of MOON [5] show for the geodesic toroidal crossing number, that

$$\overline{cr}_1(K_n) \le \frac{5}{18}\binom{n}{4}. \tag{7}$$

GERHARD RINGEL (Varenna, 1966) introduced the <u>local crossing number</u>, $l(G)$ of a graph G as the minimum, over all drawings of G, of the maximum, over all arcs, number of crossings on an arc. This, and the <u>nodal crossing number</u>, $n(G)$, where the maximum, over all nodes, is of the total number of crossings on arcs incident with a node, generalize to other orientable surfaces, and it can be shown that

$$\frac{11}{102}\binom{n-2}{2} \le l(K_n) \le \frac{1}{2}\left[\frac{1}{2}(n-1)\right]\left[\frac{1}{2}(n-3)\right], \tag{8}$$

$$\frac{4}{105}\binom{n-2}{2} \le l_1(K_n) \le \frac{5}{12}(n+1)(n-5) \ , \tag{9}$$

$$\frac{4}{35}\binom{n-1}{3} \le n_1(K_n) \le \frac{59}{216}\binom{n-2}{3} \ . \tag{10}$$

KAINEN's work [4] suggests that progress might be made by attacking the corresponding problems for the complete bipartite graph.

Zusatz während des Druckes:
In einem Brief vom 16. April 1968 teilt Professor GUY mit, daß der Satz

"However, KAINEN [4] has recently shown that

$$\lim_{n \to \infty} cr(K_n)/n^4 = \frac{1}{64} \text{"} \tag{3}$$

nicht korrekt ist. Die Arbeit [4] wurde, soweit Professor GUY bekannt ist, inzwischen zurückgezogen.

SACHS

References

[1] Guy, R.K.: A combinatorial problem, Nabla (Bull. Malayan Math. Soc.), 7 (1960), 68-72.

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[4] Kainen, P.: On a problem of Erdős, (submitted to) J. Comb. Theory.

[5] Moon, J.W.: On the distribution of crossings in random complete graphs, SIAM Journal, 13 (1965), 506-510.