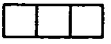
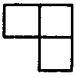
The cell growth problem and its attempted solutions

Frank Harary

Three Cell Growth Problems

Once upon a time there was an animal consisting of one square cell, . By the usual process of mitosis, it grew to be the two-celled animal, . At this stage, there were two possibilities for growth to a three-celled animal, namely  or .

It is easy to verify that there are just five different animals with four cells, and Figure 1 displays these as well as the smaller so-called "tetra-animals."

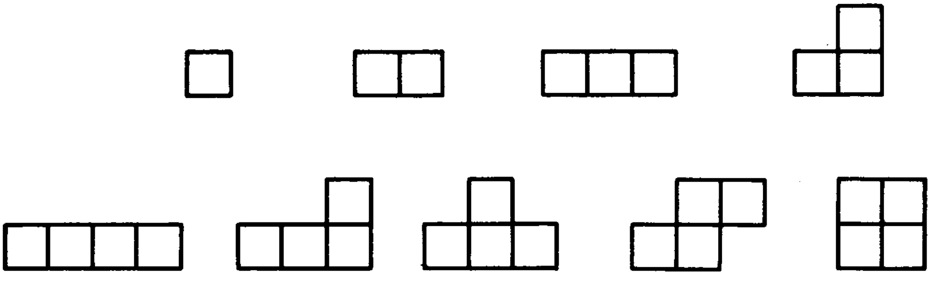


Fig. 1 The smallest tetra-animals

When there were no other kinds of animals loitering about, these square celled animals were called simply "animals". The determination of the number of simply connected tetra-animals was listed as one of 27 unsolved problems in graphical enumeration in each of the three works [1, 2, 3]. However, there are two other kinds of plane animals corresponding to the other two plane-filling regular polygons, namely, the triangle and hexagon. For this reason, we will refer to these three different counting problems as Problems IV, III, and VI.

Let us denote by a_n^{IV} the number of simply connected tetra-animals with n cells and let

$$a^{IV}(x) = \sum_{n=1} a_n^{IV} x^n. \quad \dots \quad . \quad \dots \quad .$$

Then we see from Figure 1 that

$$a^{IV}(x) = x + x^2 + 2x^3 + 5x^4 + \dots \quad .$$

Several additional coefficients are known, as we shall see later.

We now show the diagrams of the smallest triangular celled animals, or more briefly "tri-animals."

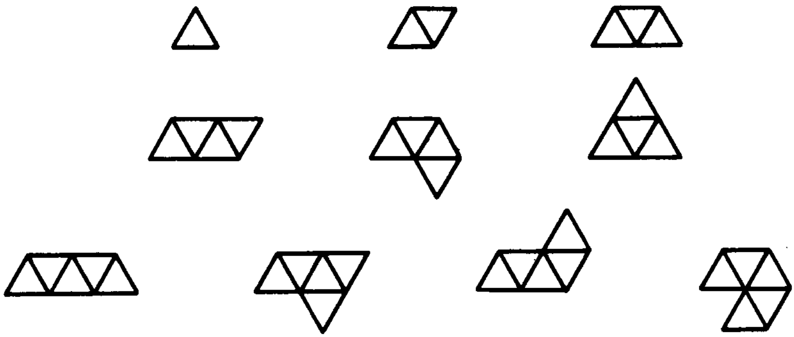


Fig. 2 The smallest tri-animals

Note that every triangular cell in a triangular animal is an equilateral triangle which can be regarded as having a unit side without loss of generality. Making the corresponding definition, for the generating function $a^{III}(x)$, we see from Figure 2 that

$$a^{III}(x) = x + x^2 + x^3 + 3x^4 + 4x^5 + \dots$$

Similarly, to the previous two figures we show the smallest hexagonal animals or "hexanimals" and write the corresponding generating function:

$$a^{VI}(x) = x + x^2 + 3x^3 + 7x^4 + \dots$$

GOLOMB [2] has made a most extensive study of tetra-animals under the name "polyominoes" and he refers to those with n square cells as n-ominoes, modifying the word "dominoes" to get this terminology.

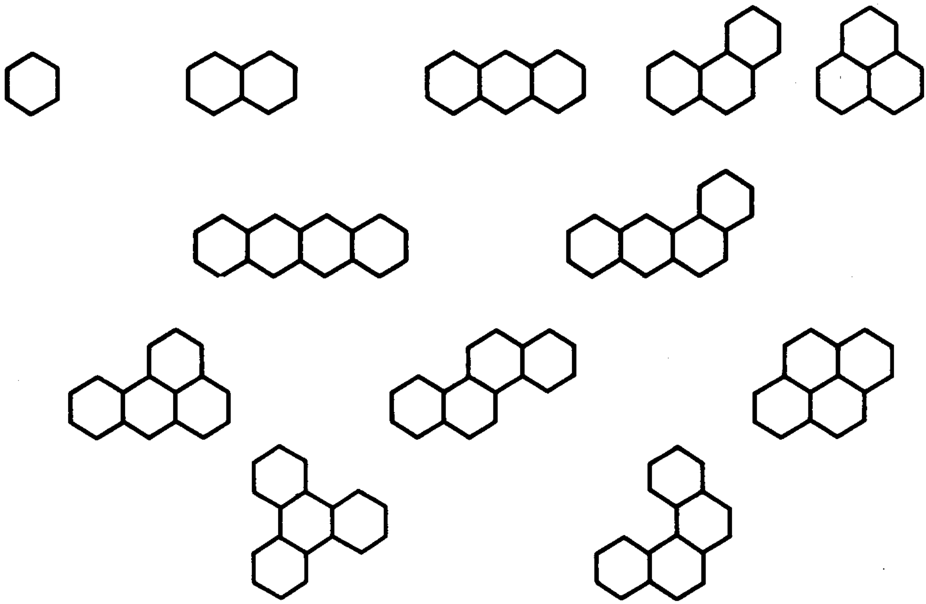


Fig. 3 The smallest hexanimals

Following an admirable custom frequently employed by our distinguished colleague, PAUL ERDŐS, I am happy to offer \$ 100 in United States currency for an exact solution for $a^{IV}(x)$. Of course, anyone who could develop methods to obtain this power series should also succeed in modifying it to get expressions for $a^{III}(x)$ and $a^{VI}(x)$ without further financial inducement. It would also be interesting to derive generating functions for those animals which are not necessarily simply connected.

In the language of graph theory, a precise definition of tetra-animals can be given as follows. A <u>tetra-animal</u> is a connected plane graph with no cut-points, in which every interior region is a unit square. Precise definitions of a <u>tri-animal</u> and a <u>hex-animal</u> are given analogously, with the stipulation that every triangle and every hexagon, respectively, has unit sides.

The calculation of lower and upper bounds for the number of animals of each kind will not be regarded as constituting a solution to this counting problem. Only a closed expression or a recursion relation which can be used to calculate the exact number of animals of each of the three types will be regarded as satisfactory. The following attempts to make progress toward the solutions of this problem will be listed in no particular order.

Attempt I - Computer Approach

In the summer of 1959 I visited the Los Alamos Scientific Laboratory and there my genial host, STAN ULAM, asked me to try to teach their computer how to think. When I stated that I didn't know anything about computers, he replied that this qualified me for the task. I suggested that their machine try to "think" about counting animals. A very primitive computer program was written which took account of the fact that the group of symmetries of a square is the dihedral group of order 8. Thus, each animal was rotated, reflected, and compared with those tetra-animals already stored, before being added to the list.

Ten men wagered $1 each on the outcome of the computer which would give the number of tetra-animals with n-cells where $n = 7, 8$, and 9. With uncanny intuition, ULAM always won, coming within three animals of the correct number.
As reported in [5] and [3], the following table contains all the known values of the number of tetra-animals a_n^{IV} and, also, of the number $\bar{a}_n^{IV}$, which includes those which are not necessarily simply connected.

Table IV

n	1	2	3	4	5	6	7	8	9	10	11	12
a_n^{IV}	1	1	2	5	12	35	107	363	1248	4271	-	-
$\bar{a}_n^{IV}$	1	1	2	5	12	35	108	369	1285	4655	17,073	63,600

KLARNER [5] proved by elegant analytic methods that when n is sufficiently large,

$$a_n^{IV} > 3.72^n$$

and

$$a_n^{VI} > 4^n.$$

When the only animals at hand are the three standard kinds, the number with n cells could be denoted t_n, s_n and h_n, thus avoiding the Roman numeral superscripts III, IV, and VI respectively.

The hero of these calculations, who guarantees the validity of Table IV is THOMAS PARKIN of the Aerospace Corporation. Although it was originally hoped that the comilation of these numbers would provide empirical data which would help the intuition in formulating a theoretical approach, this has not taken place as yet and shows no promise of doing so.

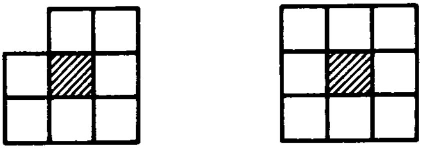


Fig. 4 Two holey tetra-animals

The smallest tetra-animal which is not simply connected has seven cells, and is shown in Figur 4 along with one of the six non-simply connected eight-celled animals, in accordance with Table IV (369 - 363 = 6).

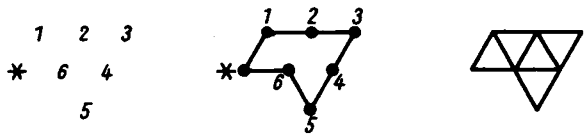
A computer program for the listing of triangular animals was developed by Dr. FOXLEY at the University of Nottingham. In the printout an asterisk (*) is used to indicate both the starting and finishing points, and the digits 1, 2, 3, etc. are used to trace out the shell of the animal.

More precise is the rectangular <u>hull</u> of a tetra-animal, or the smallest rectangle in the plane which contains all its cells. If this rectangle is $l \times w$ and $l \geq w$, then the animal is said to have <u>length</u> l and <u>width</u> w. There are two cases to be considered either $l = w$ or $l > w$. If l and w are equal, then the symmetry group of the hull itself is the dihedral group of order 8. If $l > w$, then the symmetry group of the hull is the Klein four group.

Although it would seem that these observations concerning the group of the hull might be very useful in enumerating animals, up to now they have only served to provide another unsuccessful attempt.

Attempt 2 - Triangulation of a Polygon

The tri-animal shown in Figure 5 can be redrawn as a triangulation of a 7-gon, as shown in Figure 6.



Print out Shell Tri-animal

Fig. 5 Printing a tri-animal

In Figure 6 we have a 7-gon with four additional diagonals added in such a way that every interior face is a triangle. Every triangular animal will be such a triangulation of a polygon because it will sometimes be necessary to add additional points to the interior of the polygon in order to obtain a particular tri-animal. Such a case is shown in Figure 7 where we have a hexagon with just one interior point added.

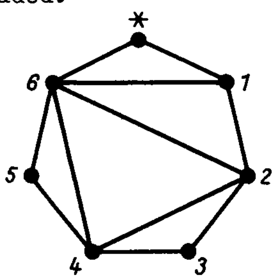


Fig. 6 A triangulation of a 7-gon

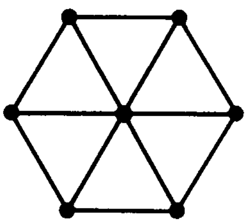


Fig. 7 A hex-animal with one interior vertex

Thus, in general, there will be two different perameters which are associated with every tri-animal, namely, the length of the exterior cycle when it is regarded as a graph and the number of interior vertices.

Such counting results go back originally to EULER himself who solved this problem without the additional interior vertices. The general solution with interior vertices permitted, was first obtained by BROWN [6]. An alternative solution was investigated by HARARY and PALMER [7].

Although the solution has now been obtained for the number of different triangulations for a polygon with interior vertices, this does not solve the original problem of obtaining the number of different tri-animals with n triangular cells. The reason is that every tri-animal is such a triangulation of a polygon, but not conversely!

One example serves to prove that the converse does not always hold and this is provided by a 7-gon with one interior vertex (see Figure 8).

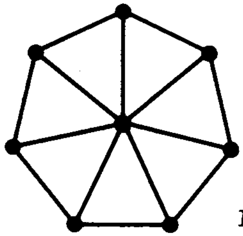


Fig. 8 A 7-gon with one interior vertex

The reason is, of course, that the triangles in Figure 8 are not equilateral and the degree of the interior vertex is greater than the maximum degree of a vertex of a tri-animal.

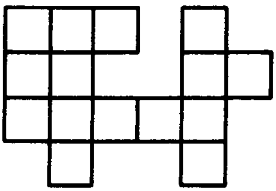
The triangulation of a square with one interior vertex also does not represent a tri-animal for its triangles are not equilateral and no interior vertices of a tri-animal can have any degree other than six.

Of course, analogous formulations can be made for the other two species of animals: the tetra-animal and the hex-animal. This would involve the subdivision of a polygon into quadrilaterals and hexagons, respectively. If the quadrilaterals can be drawn as unit squares, then the subdivision of a polygon represents a tetra-animal and, similarly, the hexagon must be regular and of the same size for a hexagon subdivision of a polygon to represent a hexagon.

Both of these subdivision problems can be solved using subtle combinatorial methods similar to BROWNS's [6] for the tri-animals and, in fact, he has done so.

Attempt 3 - Matrix Approach

To every tetra-animal there corresponds a binary matrix in the manner illustrated in Figure 9.



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Fig. 9 The matrix of a tetra-animal

The size of this animal is said to be 4 by 6. If it is stipulated that the number of rows does not exceed the number of columns, then there are still three other binary matrices which would give the animal shown in Figure 9. These would be obtained by reflecting the matrix of Figure 9 in a right-left manner and then reflecting both animals now present in the up-down fashion.

Attempt 4 - Toroidal Animals

Consider a tetra-animal of size l by w as described in Attempt 3. Consider this animal as drawn on a toroidal surface so that the right and left sides of the rectangular hull are identified, and also the top and bottom sides of the hull are identified with each other. Consider the animal A of Figure 10. If the left column of 3 cells is transported to the right, then the animal A_1 of Figure 10 is obtained. If the left column of A_1 is then moved over to the right, the result will not be an animal, since it will not even be connected. On the other hand, if the bottom row of A_1 is placed on top, then the animal A_2, shown in Figure 10, results.

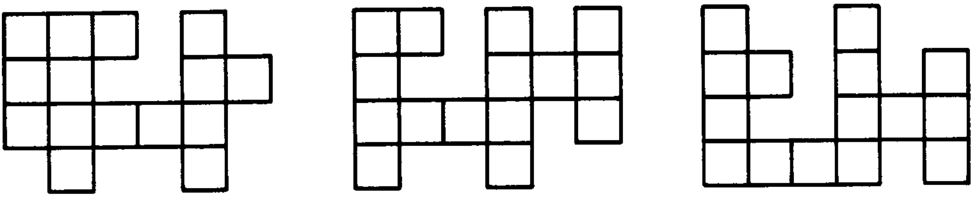


Fig. 10

These three animals, A, A_1, and A_2 are different as animals, but are regarded as equivalent "toroidal animals", or as animals drawn on a torus. Similarly, one can define cylindrical equivalence for animals (with respect to either its length or its width, for those animals which do not have a square hull). It has also been suggested that MÖBIUS equivalence of animals can be defined, etc. It is interesting to note that it is the requirement that the result of a toroidal transformation again be an animal which makes this enumeration problem difficult. If the question is altered to considering the toroidal equivalence of binary matrices (matrices with entries 0 and 1) then this new problem (which gives an upper bound) is routinely solved in the usual way with the help of the classical enumeration theorem of PÓLYA [8].

The present permutation group theoretic terminology and notation follows that in the book [9]. Let D_n be the dihedral group of degree n, and let $D_m \times D_n$ denote the cartesian product (or simply the product) of these two permutation groups. Furthermore, let $[S_2]^{D_n}$ denote the exponentiation of the symmetric group S_2 of degree 2 to the group D_n, as introduced in [10].

<u>Theorem:</u> The number of toroidal different binary matrices of size $m \times n$ is

$$Z(D_m \times D_n, 2)$$

that is, the number obtained by substituting 2 for each variable in the cycle index of the product $D_m \times D_n$.

Theorem: The number of toroidally different binary matrices of size $n \times n$ is

$$Z([S_2]^{D_n}, 2)$$

that is, the number obtained by substituting 2 for each variable in the cycle index of the exponentiation group $[S_2]^{D_n}$.

Attempt 5 - Generalized Plane Animals

It is clear why the discussion has concentrated on obtaining formulae for

$$a^{III}(x) \ , \quad a^{IV}(x) \ , \quad \text{and} \quad a^{VI}(x) \ .$$

However, it is possible to define an animal problem for other polygonal cells than triangles, squares and hexagons. Of course the result will not be a plane-filling type of animal. But with suitable restrictions on the degrees of the points, rather analogous problems can be formulated. Thus we can speak of the problem of determining the generating functions

$$a^{V}(x), \quad a^{VII}(x), \quad a^{VIII}(x), \quad \text{etc.}$$

It is correct to say that absolutely nothing is known about this further generalization of the animal problem.

Attempt 6 - Solid Animals

Very often, in mathematics, a problem becomes slightly less unsolvable when it is generalized. A classical case of this phenomenon occurs in CAYLEY's solution to the enumeration of trees corresponding to hydrocarbons, when he generalized the problem to count all the trees. Thus one is led naturally to consider animals in three dimensions (and possibly in higher dimensions if the reader is so inclined).

The first few animals whose cells are unit cubes are shown in Figure 11.

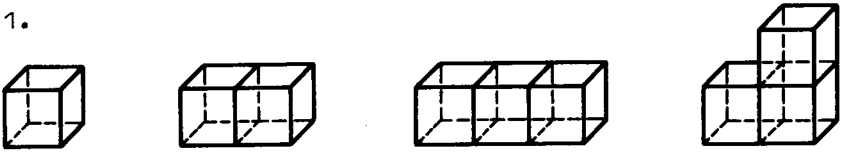


Fig. 11 The smallest solid animals

This particular approach to solving the original animal problem seems to be entirely inappropriate. The resulting structures appear even less manageable than the animals drawn in the plane. Incidentally, there are no other space-filling regular polyhedra than the cube. According to experts on packing problems, there are other polyhedra which fill all but a very small percent of 3-dimensional spaces, but this does not quite lead to a proper formulation of another 3-dimensional animal problem.

Attempt 7 - Paving Problems

This is the topic which most of the excellent interesting book bv GOLOMB [4] investigates. The question is how to pave a given rectangle or other area with certain specified animals. It is easy to verify that no rectangle, either 2×10, or 4×5, can be paved, using each of the five tetra-animals with four square cells exactly once. However, there are several ways to pave a 5×5 rectangle using all five of these animals and just one "pentominoe," cf. [2, p. 38].

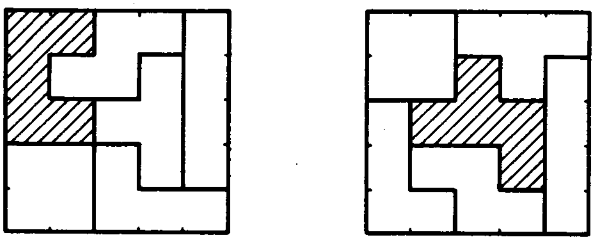


Fig. 12 Two pavings of a 5x5 square

Another context in which these problems appear is the Ising problem which is well known in theoretical physics as a model for statistical mechanics and related phenomena. Although the Ising problem can be formulated in terms of these various animal problems, as, for example, in the book [11], the animal counting problem seems to be even more subtle than the Ising problem. Certainly it is for two-dimensions, for the Ising problem has been solved but the animal problem has not.

Conclusion:

It appears very unlikely that this problem will be completely solved soon. Incidentally, it is to be hoped that attempts to solve it will lead to other interesting observations.

References

[1] Harary, F.: Unsolved problems in the enumeration of graphs. Publ. Math. Inst. Hung. Acad. Sci. 5 (1960) 63-95.

[2] Harary, F.: Combinatorial problems in graphical enumeration. Applied Combinatorial Mathematics (E.F. Beckenbach, ed.). Wiley, New York, 1964, 185-217.

[3] Harary, F.: Graphical enumeration problems. Chapter 1 in Graph Theory and Theoretical Physics (F. Harary, ed.), Academic Press, London, 1967, 1-41.

[4] Golomb, S.: Polyominoes. Scribner's, New York, 1965.

[5] Klarner, D.: Cell growth problems. Canad. J. Math. 19 (1967) 851-863.

[6] Brown, W.G.: Enumeration of triangulations of the disk. Proc. London Math. Soc. 14 (1964) 746-768.

[7] Harary, F.; Palmer, E.M.: On acyclic simplicial complexes, Mathematika, 15 (1968), to appear.

[8] Pólya, G.: Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen, Acta Math. 68 (1937) 145-254.

[9] Harary, F.: A Seminar on Graph Theory, Holt, Rinehart and Winston, New York, 1967, 1-41.

[10] Harary, F.: On the number of bicolored graphs. Pacific J. Math. 8 (1958) 743-755.

[11] Harary, F.: editor, Graph Theory and Theoretical Physics, Academic Press, London, 1967.