Some recent results on spectral properties of graphs*)

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1. Introduction. Let G be an undirected graph, without loops and without multiple edges, on a finite number of vertices. Denote by $A = A(G)$ the adjacency matrix of G, i.e.,

$$A = (a_{ij}) = \begin{cases} 1 & \text{if vertex } i \text{ and vertex } j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\lambda_1 \geq \ldots \geq \lambda_n = \lambda(G)$ the eigenvalues of A (which we now call the spectrum of G).

The general question with which this lecture deals is to examine the interplay between spectral and graphical properties of G. This question seems to have been first raised in [1], and (as the references which form only a partial bibliography attest) has been intensively studied during the last seven or eight years. I confine my discussion here to as yet unpublished work on line graphs and on imbedding questions, done jointly with D.K. RAY-CHAUDHURI and A.M. OSTROWSKI , respectively; and I will review some unsolved problems in these areas these new results suggest.

2. Line Graphs. If G is a graph, we denote by $L(G)$ - the line graph of G - the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent if and only if the corresponding edges of G have exactly one common vertex.

Lemma: $\lambda(L(G)) \geq -2$. If G has more edges than vertices, $\lambda(L(G)) = -2$.

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Proof: Let K be the (0,1) matrix whose rows correspond to the vertices of G, and whose columns correspond to the edges of G, with each column containing exactly two 1's, in the rows whose corresponding vertices are joined by the edge corresponding to the column. Clearly,

$$K^T K = 2I + A(L(G)).$$

Since $K^T K$ is positive semi-definite, $\lambda(L(G)) \geq -2$. If G has more edges than vertices, $K^T K$ is singular, so 0 is its least eigenvalue, and $\lambda(L(G)) = -2$.

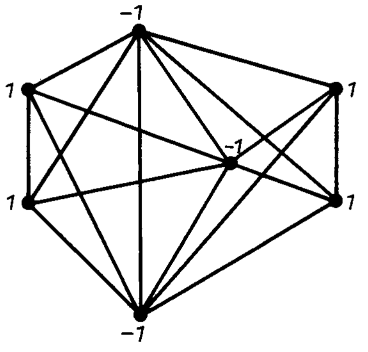
The property stated in the lemma is so striking that one is tempted to conjecture that it characterizes line graphs. In this connection, Ray-Chaudhuri has recently proved that if every vertex of G has valence at least 44, $\lambda(G) \geq -2$, and if G has the further property that, if i and j are any adjacent vertices, there are at least two vertices adjacent to i but not to j, then G is a line graph. The number 44 is not best possible, and it would be interesting to find the smallest possible number for the minimum valence for which the theorem remains true.

If we assume G regular (i.e., every vertex has the same valence), then previous work ([2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]) on particular families of line graphs has shown that if we assume G regular and connected, $\lambda(G) \geq -2$, it is still not necessarily true that G is a line graph. The reason is that the work cited has turned up a finite list of exceptions. But even the conjecture that a regular connected graph G with $\lambda(G) \geq -2$ of sufficiently high valence must be a line graph is also false. For let H_n be the graph formed by deleting a 1-factor from the complete graph on 2n vertices. Then $\lambda(H_n) = -2$ is easily seen, since $A(H_n)$ consists of a matrix of order 2n with all entries 1 except for 2×2 blocks of zeros on the diagonal. Now when $n \geq 4$, H_n is not a line graph. The reason is that the number of vertices adjacent to each of two nonadjacent vertices in H_n is $2n - 2$ (≥ 6 if $n \geq 4$). But, in a line graph, there are at most four vertices adjacent to each of two nonadjacent vertices.

Theorem: Let G be a regular connected graph of valence at least 17, $\lambda(G) \geq -2$. Then $G \cong H_n$ for some n or G is a line graph.

The proof is long, but we indicate its spirit (i.e., the translation of eigenvalue information into graph information) by the following lemma, which is used in the proof:

The graph

$H =$ 

cannot be an induced subgraph of a regular graph G with $\lambda(G) \geq -2$.

The reason is as follows: Suppose $H \subset G$. Then $A(H)$ would be a principal submatrix of $A(G)$. The vector $x = (1,1,-1,-1,-1,1,1)$ indicated on the diagram for H satisfies $A(H)x = -2x$. Therefore, the vector $y = (x; 0,\dots,0)$ satisfies $(A(G)y,y)/(y,y) = -2$. By the well-known characterization of the minimum eigenvalue of a real symmetric matrix, this implies y is an eigenvector of $A(G)$, corresponding to the eigenvalue -2. Since G is regular, $A(G)u = du$, where d is the valence of G, and $u = (1,\dots,1)$. Because $d \neq -2$, and $A(G)$ is symmetric, $(u,y) = 0$. But this means that the sum of the co-ordinates of x is 0, which is false.

It is worth noting that the number 17 is best possible. The regular graph of valence 16 on 27 vertices described in [15] satisfies $\lambda(G) = -2$, but it is neither a line graph nor H_n.

3. Related Unsolved Problems. The basic question is the enumeration of all graphs G with $\lambda(G) \geq -2$. In the case of regular graphs, it seems reasonable to conjecture that there are only a finite number apart from the H_n series, but we do not yet have the tools to show this. For the not necessarily regular graphs, there is at the start the problem of finding the smallest minimum valency (to replace 44) in Ray-Chaudhuri's theorem.

4. Imbedding. The problem considered here is to calculate two invariants of a given graph G. We use the notation $d(G)$ for the minimum valence of the vertices of G.

$$\mu(G) = \lim_{d \to \infty} \sup_{\substack{H \supset G \\ d(H) > d}} \lambda(H)$$

$$\mu_R(G) = \lim_{d \to \infty} \sup_{\substack{H \supset G \\ H \text{ regular} \\ d(H) > d}} \lambda(H)$$

The problem of greater interest is to know $\mu_R(G)$, and this seems hard. But $\mu(G)$ can be calculated in the following way: let Γ be the set of $(0,1)$ matrices C in which the number of rows is the same as the number of vertices of G, in which each row of C contains at least one 1, but deleting any column of C destroys this property. Then

$$\mu(G) = \max_{C \in \Gamma} (\text{least eigenvalue of } A - CC^T). \qquad (4.1)$$

The proof runs as follows. Let H be a graph containing G and several cliques. Each clique corresponds to a column of C, and the clique corresponding to a given column has each vertex in it adjacent to the vertices of G corresponding to the rows of C where that column has a 1. If each clique has d vertices, then H is a graph in which each vertex has valence at least d and H contains G. Letting $d \to \infty$ and applying the theorem of ROUCHÉ-HURWITZ , we verify that the left side of (4.1) is at least as great as the right side. To prove it is no greater, we assume that $H \supset G$, $\lambda(H)$ bounded from below. Then if $d(H)$ is large enough, it can be shown that H contains a subgraph corresponding to a matrix C in the manner described above.

5. Related Unsolved Problems. One can use (4.1) to prove the following: let $c(G)$ be the maximum n such that $K_{1,n} \subset G$. Then for $\alpha \geq 1$,

$$\lim_{d \to \infty} \max_{\substack{d(G) > d \\ \lambda(G) \geq -\alpha}} c(G) = [\alpha]^2 + [(\alpha-1)(\alpha-[\alpha])]. \tag{5.1}$$

Frank HARARY has raised the more general question: let $c_m(G)$ be the maximum n such that $K_{m,n} \subset G$. Let $\alpha \geq -1$ be given. What is

$$\lim_{d \to \infty} \max_{\substack{d(G) > d \\ d(G) \geq -\alpha}} c_m(G) \ ?$$

The problem of a formula for $\mu_R(G)$ appears more difficult, but perhaps useful bounds can be obtained. It is, in particular, attractive to conjecture that, for $\alpha \geq 1$,

$$\lim_{d \to \infty} \max_{\substack{d(G) > d \\ \lambda(G) \geq -\alpha \\ G \text{ regular}}} c(G) \overset{?}{=} [\alpha] .$$

But this has only been proved for $1 \leq \alpha \leq 2$.

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