Graphs and set systems

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A graph is an ordered pair $\langle g, G \rangle$ where G is a system of pairs of the finite set g. Thus the notion of (finite) set-systems is a natural generalization of the notion of graphs: a set-system is an ordered pair $\langle h, H \rangle$ where h is a finite set (the set of vertices) and H is a system of subsets (edges) of h. Here we do not allow multiple edges, but the graphs may have such edges.

The aim of this lecture is to examine problems arising if we want to generalize notions and theorems of graph theory for set-systems.

1. Let $\mathfrak{H} = \langle h, H \rangle$ be a set system. Denote by K_E the complete graph spanned by the elements of the set E and put $\mathfrak{G}_{\mathfrak{H}} = \bigcup_{E \in H} K_E$ (common edges with respective multiplicities). A circuit of $\mathfrak{G}_{\mathfrak{H}}$ is said to be a circuit of $\mathfrak{H}$. Such a circuit is trivial, if all of its edges belong to the same K_E. Among the non-trivial circuits the most interesting ones are the simple circuits, the edges of which belong to different K_E-s. Note that a trivial and a non-trivial circuit may have the same sequence of vertices.
It is obvious, that

Theorem 1: A shortest non-trivial circuit is simple.

If $\langle h, H \rangle$ and $\langle h', H' \rangle$ are two set-systems and $h' \subset h$, $H' \subset H$ then we shall say that $\langle h', H' \rangle$ is a subsystem of $\langle h, H \rangle$.
A set-system is a uniform k-system, if its edges are k-tuples.
A graph is a uniform 2-system.
The notion of chromatic number was generalized for set-systems by ERDŐS and HAJNAL [1] [1]. The chromatic number of the set-system

1) The property that a set-system has chromatic number 2 was introduced by Miller (property B).

$\langle h, H \rangle$ is the least natural number n of the following property: there exists a decomposition of h into classes $A_1, \ldots, A_n$ such that none of these classes contains an edge of the set-system [2]. The set $\{A_1, \ldots, A_n\}$ is a <u>colouring</u> of $\langle h, H \rangle$.

A notion of graph theory may have more possible generalizations. A forest is a graph containing no (non-trivial) circuits. A set-system of this property will be said to be <u>circuitless</u>. A graph is a forest if and only if $v = e + \nu$, where v, e and ν are the number of vertices, edges and connected components of the graph, resp. Let ν denote the number of connected components of $\mathfrak{G}_{\mathfrak{H}}$, then

<u>Theorem 2</u>: A set-system $\langle h, H \rangle$ is circuitless if and only if

$$|h| = \nu + \sum_{E \in H} (|E| - 1) \tag{1}$$

The proof is just like the proof of (2.5) in [2].

ERDŐS suggested the question: what can we say about set-systems which contain simple circuits only of length 2 ? Graphs of this property are forests.

<u>Theorem 3</u>: Let $\mathfrak{H} = \langle h, H \rangle$ be a set-system having simple circuits only of length 2; suppose that two edges of $\mathfrak{H}$ have at most two points in common. Let further $\mathfrak{G}_{\mathfrak{H}}$ have ν connected components and μ lobes (a cut-edge is also a lobe) then

$$\sum_{E \in H} (|E| - 2) + \mu + \nu = |h|. \tag{2}$$

P r o o f : We use induction on $|h|$; for $|h| = 1$ the theorem is obvious.

Let $|h| \geq 2$, $H = \{E_1, \ldots, E_l\}$

If $\mu > 1$, then (2) holds for every lobe by the induction hypothesis and then adding these equalities we obtain (2).

Thus we may suppose that $\mu = 1$ and consequently $\nu = 1$. If $x \in h$ and x is incident say only to E_1 then putting

2) Speaking about the chromatic number of a set-system - and only in this case - we suppose that it does not contain edges of one element.

$h_1 = h - \{x\}$, $H_1 = \{E_1 - \{x\}, E_2, \ldots, E_l\}$, the application of the induction hypothesis on $\langle h_1, H_1 \rangle$ proves the theorem.

Suppose now that such an x does not exist and consider the sets $E \cap F$ $(E, F \in H,\ E \cap F \neq \emptyset,\ E \neq F)$.

a) If $E, F \in H$, $E \neq F$ then $|E \cap F| \neq 1$ (i.e. $|E \cap F| = 0$ or 2).

Suppose that $E \cap F = \{a\}$. Because of $\mu = 1$ there exists a shortest path $(b_o, \ldots, b_s)$ $(b_i \neq a)$ of $\mathfrak{G}_{\mathfrak{H}}$ connecting a vertex of E to a vertex of F. Then $(a, b_o, \ldots, b_s)$ span a simple circuit of length ≥ 3, this is a contradiction.

b) If $E \cap F \neq E' \cap F'$ $(E \neq F\ ;\ E' \neq F';\ E, E', F, F' \in H)$, then $E \cap F \cap E' \cap F' = \emptyset$.

Suppose that $x \in E \cap F \cap E' \cap F'$. Because of a) $E \cap F = \{x,y\}$, $E' \cap F' = \{x,z\}$. If say $E' = E$, then putting $F \cap F' = \{x,t\}$, (y,z,t) span a simple circuit. If $E \neq E'$, $E \neq F'$, $F \neq E'$, $F \neq F'$ then putting $E \cap E' = \{x,t\}$, $F \cap F' = \{x,u\}$, (t,z,u,y) span a simple circuit.

We construct a pair graph $\mathfrak{G}_o$ as follows. Let $A = \{E \cap F:$ $E \neq F \in H,\ E \cap F \neq 0\}$ and let the set of vertices of $\mathfrak{G}_o$ be $A \cup H$. We connect $U \in A$ to $V \in H$ if and only if $U \subset V$.

$\mathfrak{G}_o$ is a tree; really, because of $\nu = 1$ it is connected, and if it contained a circuit $(E_1 \cap F_1, E_1', \ldots, E_r \cap F_r, E_r')$ then obviously $r \geq 3$ and taking $x_i \in E_i \cap F_i$ $(1 \leq i \leq r)$ $(x_1, \ldots, x_r)$ would be a simple circuit of length r.

By a) and b) $\mathfrak{G}_o$ has $|H| + \frac{|h|}{2}$ vertices and $\sum_{E \in H} \frac{|E|}{2}$ edges. Consequently

$$\frac{|h|}{2} + |H| = \sum_{E \in H} \frac{|E|}{2} + 1,$$

which is equivalent with (2).

This theorem implies the following conjecture of ERDŐS:

> <u>Theorem 4:</u> If H is a system of triplets of h and $|H| \geq |h| - 1$ then $\langle h, H \rangle$ contains a simple circuit of length ≥ 3.

A third possibility to generalize the notion of forest is the following: let the set-system $\langle h, H \rangle$ be said to be a <u>forest</u> if for any $\langle h', H' \rangle$ subsystem of it we have $|h'| \geq |H'| + 1$. A <u>tree</u> is a forest for which $|h| = |H| + 1$.

ERDŐS conjectured, that

> <u>Theorem 5:</u> A forest has chromatic number 2.

He mentioned, that his conjecture is sharp for uniform 3-systems: Let the set of vertices of a set-system be the points of the projective plane of 7 points (see Fig.) and let the edges be the triplets of points lying on a line. For this set-system it holds, that any subset $\langle h', H' \rangle$ of it satisfies $|h'| \geq |H'| + 1$, except when $h' = h$, $|h| = |H|$, although it has chromatic number 3. For uniform k-systems ($k \geq 4$) the theorem is probably not sharp.

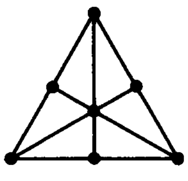

Abb. 1

P r o o f : We use induction on $|h|$. For $|h| = 1$ the theorem is obvious. Suppose that the theorem is true if $|h| < k$ $(k \geq 2)$. Let $\langle h_1, H_1 \rangle$ be a maximal tree contained in the forest $\langle h, H \rangle$ $(|h| = k)$, such that $h_1 \neq h$. Put $h_2 = h - h_1$, $H_2 = H - H_1$. If there is no edge having a common point with both of h_1 and h_2, then $\langle h_2, H_2 \rangle$ is a forest and by the induction hypothesis we can colour $\langle h_1, H_1 \rangle$ and $\langle h_2, H_2 \rangle$ by 2 colours.

Now let $E_o \in H$, $x \in E_o \cap h_1$, $y \in E_o \cap h_2$. Put $H_2' = \{E \cap h_2 : E \in H_2, E \neq E_o\}$. $\langle h_2, H_2 \rangle$ is a forest; really,

$|h_2| = |h| - |h_1| \geq |H| + 1 - |H_1| - 1 = |H_2| \geq |H_2'| + 1$ and if $\langle h_3, H_3 \rangle$ is a subsystem of it $(h_3 \neq h_2)$, then by the maximality of $\langle h_1, H_1 \rangle$ we obtain $|h_1 \cup h_3| > |H_1 \cup H_3| + 1$ and thus $|h_3| \geq |H_1| + |H_3| - |h_1| + 2 = |H_3| + 1$.

Let (A,B) be a colouring of $\langle h_1, H_1 \rangle$, (C,D) a colouring of $\langle h_2, H_2' \rangle$. Let $x \in A$, $y \in C$, say. Then $(A \cup D, B \cup D)$ is a colouring of $\langle h, H \rangle$. The proof is complete.

2. TUTTE and ZYKOV proved that there exist graphs of arbitrary large chromatic number containing no triangles. ([3], [4]).
This theorem was generalized by ERDŐS. He showed that for any n and s there exists a graph of chromatic number n containing circuits only longer than s. His proof uses the "probabilistic method", i. e. it does not give a concrete example. A concrete example was given by ZYKOV for $s = 5$ and by NEŠETŘIL for $s = 6, 7$.([5]).

ERDŐS and HAJNAL generalized the theorem of ERDŐS for set-systems also by probabilistic methods. We prove the theorem of ERDŐS and HAJNAL by a direct construction. This gives a construction for ERDŐS' theorem. ([2])

Theorem 6: Let k, n, s be given natural numbers. We construct a uniform k-system of the following properties:

(i) its non-trivial circuits are longer than s;
(ii) it has chromatic number n.

In fact, instead of (i) ERDŐS and HAJNAL required an other property; the equivalence of the two properties is shown by Theorem 2.

P r o o f : Let $\mathfrak{H}(k,n,s)$ denote the required set-system.
We shall construct it by induction on n. For $n = 2$ a set-system consisting of a single k-tuple satisfies (i) and (ii).

Let us suppose that $\mathfrak{H}(k,n-1,s)$ is already constructed. Let us take some copies $\mathfrak{H}_1, \mathfrak{H}_2, \dots, \mathfrak{H}_r$ of $\mathfrak{H}(k,n-1,s)$ (r will be determined later). Put $\mathfrak{H}_i = \langle h_i, H_i \rangle$, $h = \bigcup_{i=1}^{r} h_i$, $H' \bigcup_{i=1}^{r} H_i$,

$Q = \{h_1,\dots,h_r\}$. We are going to carry out the construction by adding some further k-tuples of h. Let H" denote the set of these new k-tuples.

Let us investigate what kind of properties the set-system $\mathfrak{K} = \langle h, Q \cup H'' \rangle$ must have to guarantee that $\mathfrak{H} = \langle h, H' \cup H'' \rangle$ satisfies the requirements (i) and (ii).
Put $\bar{E} = E$ if $E \in H''$ and $\bar{E} = h_i$ if $E \in H_i$. Thus if $E \in H' \cup H''$ then $\bar{E} \in Q \cup H''$ and $E \subset \bar{E}$. To a circuit of $\mathfrak{G}_{\mathfrak{H}}$ we make correspond a circuit of $\mathfrak{G}_{\mathfrak{K}}$ such that we substitute the edges of it belonging to K_E by edges belonging to $K_{\bar{E}}$. If the original circuit has been non-trivial and the obtained one is trivial then the original one has been a circuit of $\mathfrak{H}_i$ for some i, and consequently it is longer then s. Thus if we require the non-trivial circuits of $\mathfrak{K}$ to be longer than s, (i) is satisfied.

On the other hand we want $\mathfrak{H}$ to be at least (n+1)-chromatic. Let us split the set h into n disjoint classes $A_1,\dots, A_n$. If there is an i such that $h_i \cap A_1 = \emptyset$ then by the induction hypothesis one of the classes $A_2,\dots, A_n$ contains an edge of $\mathfrak{H}_i$. Thus it is enough to guarantee that if $h_i \cap A_1 \neq \emptyset$ for every i then one of the edges of H" is a subset of A_1.

Thus it is enough to construct a set-system $\mathfrak{K}(k,l,s) = \langle h, K \rangle$ of the following properties:

(i) $K = H^* \cup Q$ where the edges of H^* and Q are k-tuples and l-tuples respectively;
(ii) if $E \in Q$ and $F \in Q$ then $E \cap F = \emptyset$
(iii) $\mathfrak{K}(k,l,s)$ contains non-trivial circuits only longer than s.
(iv) if M is a subset of h such that for any $E \in Q$ $E \cap M \neq \emptyset$ then there exists an $F \in H^*$ such that $F \subset M$.

We prove this lemma by induction on k and simultanously on s. For k = 1 and any s the set-system $\langle h, K \rangle$, $K = H^* \cup Q$, $h = \{1,\dots,l\}$, $H^* = \{\{1\},\dots, \{l\}\}$, $Q = \{h\}$ satisfies the requirements.

For $s = 1$ (iii) is insignificant. Let $E_1, \dots, E_k$ be disjoint l-tuples and let H^* consist of all k-tuples of $E_1 \cup E_2 \cup \dots \cup E_k = h$. Putting $K = H^* \cup Q$, $Q = \{E_1, \dots, E_k\}$, $\langle h, K \rangle$ satisfies the other requirements.

We may suppose that we have constructed $\mathfrak{R}(k-1,l,s)$ and $\mathfrak{R}(k_o,l,s-1)$ for any k_o. We know, that $\mathfrak{R}(k-1,l,s) = \langle h_o, Q_o \cup H_o^* \rangle$ where H_o^* consists of $(k-1)$-tuples and Q_o consists of disjoint l-tuples. Put $|H_o^*| = k_o$. Then $\mathfrak{R}(k_o,l,s-1) = \langle h_1, K_1 \rangle$ exists and $K_1 = Q_1 \cup H_1^*$ where the elements of Q_1 and H_1^* are l-tuples and k_o-tuples respectively. Let us take for every $F \in H_1^*$ a copy $\mathfrak{R}_F = \langle h_F, A_F \rangle$ of $\mathfrak{R}(k-1,l,s)$ such that $h_{F_1} \cap h_{F_2} = \emptyset$ if $F_1 \neq F_2$. Put $A_F = Q_F \cup H_F^*$ where Q_F and H_F^* are the sets of l-tuples and $(k-1)$-tuples of K_F, respectively. To every $J \in H_F^*$ we make correspond an element x_J of F, to different J - s we make correspond different x_J - s, and we denote the set $J \cup \{x_J\}$ by J^*. Put $\tilde{H}_F = \{J^* : J \in H_F^*\}$, $H^* = \cup \tilde{H}_F$, $Q = \bigcup_{F \in H_1^*} Q_F \cup Q_1$, $K = H^* \cup Q$, $h = \bigcup_{F^* \in H_1^*} h_F \cup h_1$. We shall prove, that $\mathfrak{R} = \langle h, K \rangle$ satisfies the requirements (i) - (iv).

A) (i) and (ii) are obvious.

B) Put $\bar{E} = E$ if $E \in Q_1$ and $\bar{E} = F$ if $E \in Q_F \cup \tilde{H}_F$. Let $a_1, \dots, a_t$ be the vertices of a shortest nontrivial circuit S of $\mathfrak{R}$. Because of Theorem 1, S is simple. We make correspond to S a circuit $\bar{S}$ of the same vertices substituting its edges belonging to K_E by edges of $K_{\bar{E}}$.

α) If $\bar{S}$ is trivial then obviously its edges belong to $Q_F \cup \tilde{H}_F$ for some F. $a_i \notin F$; really, $a_i \in F$ would imply that the two edges of S incident with a_i belong to the same K_E which is a contradiction since S is simple. But then $a_1, \dots, a_t$ span a nontrivial circuit of $\mathfrak{R}_F$ and thus $t > s$.

β) Suppose that $\bar{S}$ is non-trivial. Obviously there exists an a_i which belongs to h_F for some $F \in H_1^*$. The two edges of $\bar{S}$ incident with a_i belong to K_F, hence $\bar{S}$ is not simple. Because of Theorem 1, $\bar{S}$ is not a shortest non-trivial circuit of $K(k_o,l,s-1)$, i.e. $t > s$, q.e.d.

C) Let M be a subset of h such that $M \cap E$ is not empty for any $E \in Q$. Then there is an $F \in H_1^*$ such that $F \subset M$. Further there is $J \in \mathfrak{H}_H^*$ such that $J \subset M$. $x_J \in F$ and therefore $J^* = J \cup \{x_J\} \subset M$.

The proof of Theorem 6 is complete now. It is interesting, that if I wanted to tell the construction only for graphs I ought to use set-systems, too.

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