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The q-perfect Graphs. Part I.: The case $q = 2$

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ABSTRACT

Let q be a positive integer. Many graphs admit a partial coloring with q colors and a clique partition such that each of the cliques is "strongly colored", that is: contains the largest possible number of different colors. If a graph G and all its induced subgraphs have this property, we say that G is "q-perfect" (Lovász [8]). For $q = 1$, this reduces to the classical concept of perfect graph. In this paper, we study the graphs which appear to be q-perfect for $q = 2$.

1. General results

Let G be a "simple graph" (no loops, no multiple edges), of order n; we denote by $\alpha(G)$ the stability number, by $\theta(G)$ the least number of cliques which cover the vertex set, by $\omega(G)$ the maximum size of a clique, and by $\chi(G)$ the chromatic number.

Let q be a positive integer. A *partial q-coloring* of G is a set of q pairwise disjoint stable sets $S_1, S_2, \dots, S_q$, each one corresponding to a "color"; some of the vertices may have no color. The largest possible number of colored vertices in a partial q-coloring is denoted by $\alpha_q(G)$. A partial q coloring with $\alpha_q(G)$ colored vertices is said to be *optimal*.

Let $M = (C_1, C_2, \dots)$ be a partition of $V(G)$ into cliques; by definition, the *q-norm* of M is:

$$B_q(M) = \sum_{j \geq 1} \min\{|C_j|, q\}.$$

We denote $\theta_q(G)$ the minimum q-norm for the clique partitions of G. If $B_q(M) = \theta_q(G)$, we say that M is *q-optimal.*

For every clique partition $M = (C_1, C_2, \ldots,)$ and for every partial q-coloring $(S_1, S_2, \ldots, S_q)$, we have

$$|\bigcup_{i=1}^{q} S_i| = \sum_j |C_j \cap \bigcup S_i| \leq \sum_j \min\{|C_j|q\} = B_q(M) \qquad (1)$$

Hence $\alpha_q(G) \leq \theta_q(G)$. If every induced subgraph G_A of G satisfies $\alpha_q(G_A) = \theta_q(G_A)$, we say that G is *q-perfect;* clearly $\alpha_1(G) = \alpha(G)$, $\theta_1(G) = \theta(G)$, and a graph is 1-perfect if and only if it is perfect. This concept is due to Lovász [8], who noticed that "every comparability graph is q-perfect" is equivalent to a famous theorem of Greene and Kleitman [6].

Clearly, with a partial q-coloring, a clique C contain at most $\min\{|C|, q\}$ colored vertices; a clique C which contains exactly $\min\{|C|, q\}$ colored vertices is said to be *strongly colored.*

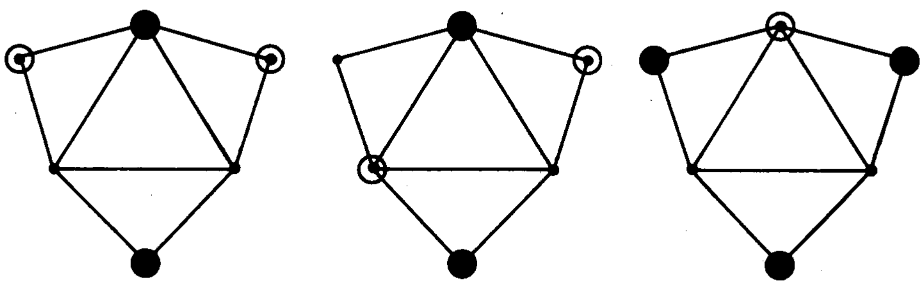


Fig. 1. *The optimal 2-colorings of the Hajós graph H, with* $\alpha_2(H) = 4$, $\theta_2(H) = 5$

Proposition 1. *Let G be a graph, and let q be a positive integer; the following conditions are equivalent:*

(i) $\alpha_q(G) = \theta_q(G)$;

(ii) *for at least one clique partition M and one partial q-coloring, all the cliques of M are strongly colored;*

(iii) *for every optimal partial q-coloring and for every q-optimal clique partition M, all the cliques of M are strongly colored.*

This proposition follows immediately from the inequalities in (1).

Proposition 2. *Let G be a graph of order n and let q be positive integer $\geq \omega(G)$. Then $\alpha_q(G) = \theta_q(G)$ if and only if $\chi(G) \leq q$.*

First, assume that $\alpha_q(G) = \theta_q(G)$, and let M be a q-optimal clique partition. We have:

$$\alpha_q(G) = B_q(M) = \sum_{C \in M} \min\{|C|, q\} = \sum_{C \in M} |C| = n.$$

So, it is possible to color G with only q colors; hence $q \geq \chi(G)$.

Conversely, if q colors suffice, we have, for a q-optimal clique partition M,

$$\alpha_q(G) = n = \sum_{C \in M} |C| = \sum_{C \in M} \min\{|C|, q\} = B_q(M) = \theta_q(G).$$

Thus $\alpha_q = \theta_q$.

Corollary. *Every graph G is q-perfect for all $q \geq \chi(G)$.*

Let $A \subset V(G)$ and let $q \geq \chi(G)$. We have: $q \geq \chi(G_A) \geq \omega(G_A)$. So, by Proposition 2, $\alpha_q(G_A) = \theta_q(G_A)$. Since this equality holds for every $A \subset V(G)$, the graph G is q-perfect.

For $q = 2$, the q-perfectness has a special meaning, since $n - \alpha_2(G)$ is the least number of vertices that we need to remove from G to obtain a bipartite graph. We shall give here more specific conditions for a graph G to be 2-perfect.

Lemma. *Let G be a partitionable graph, and let q be a positive integer $\leq \omega(G)$. Then $\alpha_q(G) = q\alpha(G)$ and $\theta_q(G) = q\alpha(G) + 1$.*

Proof. Let G be a partitionable graph with $\alpha(G) = \alpha$ and $\omega(G) = \omega$, that is: $\alpha \geq 2$, $\omega \geq 2$, and for every vertex x, there exist two partitions $P = (\{x\}, S_1, S_2, \ldots, S_\omega)$ and $M = (\{x\}, C_1, C_2, \ldots, C_\alpha)$ where the S_i's are maximum stable sets and the C_j's are maximum cliques. So $|S_i \cap C_j| = 1$ for all i, j, and $|S_i| = \alpha$, $|C_j| = \omega$ (see Bland et al., [2]).

For a partitionable graph G, we have $\alpha_q(G) \geq q\alpha$ because $(S_1, S_2, \ldots, S_q)$ is a partial q-coloring with $q\alpha$ colored vertices. We cannot have $\alpha_q(G) > q\alpha$ because this would imply that one of the q color classes be of size $> \alpha$, a contradiction. Hence $\alpha_q(G) = q\alpha$.

On the other hand, we have $\theta_q(G) \leq q\alpha + 1$ because $M = (\{x\}, C_1, C_2, \ldots, C_\alpha)$ is a clique partition with $B_q(M) = q\alpha + 1$. We cannot have $\theta_q(G) < q\alpha + 1$, because this would imply the existence of a clique partition M' with $B_q(M') \leq q\alpha$; since $\alpha\omega + 1 = |V(G)| \leq |M'|\omega$, we have $|M'| > \alpha$, and at least one class of M' is a singleton $\{x_0\}$ without changing the value of $B_q(M')$. Since G is partitionable, there exists a clique partition $\overline{M} = (\{x_0\}, \overline{C}_1, \overline{C}_2, \ldots, \overline{C}_\alpha)$ with $|\overline{C}_j| = \omega$ for all j; we have $q\alpha + 1 = B_q(\overline{M}) \leq B_q(M') \leq q\alpha$. A contradiction.

Thus, $\theta_q(G) = q\alpha + 1$. ∎

Theorem 1. *Every 2-perfect graph is perfect.*

Proof. If the 2-perfect graph G was not perfect, it would contain an induced partitionable subgraph G_A with $\alpha(G_A) = \alpha \geq 2$ and $\omega(G_A) = \omega \geq 2$ (Lovász [7], Chvátal [4]). By the lemma, $\alpha_2(G_A) \neq \theta_2(G_A)$, which contradicts that G is 2-perfect. ∎

2. Parity graphs and 2-perfectness

A chain μ is odd (resp. even) if the number of edges in μ is odd (resp. even). A chain μ is *chordless* if its vertices induce an elementary chain. A graph G is a *parity graph* if for $x, y \in V(G)$, all the chordless chains joining the vertices x and y have the same parity. For an elementary cycle, we say that two chords $[x, y]$ and $[z, t]$ *cross* if the vertices x, z, y, t are encountered in this order on the cycle. One can see that a graph is a parity graph if and only if every odd elementary cycle of length ≥ 5 has two crossing chords (Burlet and Uhry [3]). The parity graphs have been considered first by Sachs [11] who proved that every parity graph is perfect. This follows also from a more general result of Meyniel [9]. Burlet and Uhry [3] gave a polynomial recognition algorithm for a parity graph. A bipartite graph defined by two vertex-sets X and Y is a parity graph, because every chain joining $x \in X$ and $x' \in X$ is even, and every chain joining $x \in X$ and $y \in Y$ is odd. A graph such that each block is a clique is also a parity graph.

We shall show that the parity graphs are 2-perfect by using the concept of *Cartesian sum* $G+H$ of two graphs G and H: by definition, the vertex-set of $G+H$ is the Cartesian product $V(G) \times V(H)$, and two points (x, y), (x', y') are adjacent in $G+H$ if either $x = x'$ and $yy' \in E(H)$, or $xx' \in E(G)$ and $y = y'$.

Lemma 1. *$\alpha_q(G) = \alpha(G + K_q)$ for every positive integer q.*

Denote by K_q the complete graph on $\{1, 2, , \ldots, q\}$. A stable set $\overline{S}$ of $G + K_q$ defines a partial q-coloring $(S_1, S_2, \ldots, S_q)$ of G as follows: $x \in S_i$ if and only if $(x, i) \in \overline{S}$. Clearly $(S_1, S_2, \ldots, S_q)$ is a partial coloring of G, and $|\bigcup S_i| = |\overline{S}|$. Conversely, each partial q-coloring of G defines a stable set of $G + K_q$ with the same cardinality. Hence:

$$\alpha_q(G) = \alpha(G + K_q).$$

Lemma 2. *$\theta_q(G) = \theta(G + K_q)$ for every positive integer q.*

A clique partition M of G with $B_q(M) = \theta_q(G)$ can be associated with a clique partition $\overline{M}$ of $G+K_q$ as follows: each $C \in M$ with $|C| \geq q$ will be represented in $\overline{M}$ by the cliques $C \times \{i\}$ with $i = 1, 2, , \ldots, q$; each $C \in M$ with $|C| < q$ will be represented in $\overline{M}$ by the cliques $\{x\} \times \{1, 2, \ldots, q\}$ with $x \in C$. So, $\overline{M}$ is a clique partition of $G + K_q$. We have:

$$\theta(G + K_q) \leq |\overline{M}| = \sum_{C \in M} \min\{|C|, q\} = B_q(M) = \theta_q(G). \tag{2}$$

Conversely, consider a minimum clique partition $\overline{L}$ of $G + K_q$. First replace in $\overline{L}$ each clique whose projection on G is a singleton $\{x\}$ by a *vertical clique* $\{x\} \times \{1, 2, \ldots, q\}$ and replace each *horizontal clique* $C \times \{i\}$ by $(C - \{x\}) \times \{i\}$. This does not change the cardinality of the clique partition, that we shall also denote by $\overline{L}$.

In $\overline{L}$, the horizontal cliques whose projection on K_q is $\{1\}$ are as numerous as the horizontal cliques whose projection is $\{i\}$, where $i > 1$ (otherwise $\overline{L}$ would not be minimum). Denote by P_1 the set of all C such that $C \times \{1\} \in \overline{L}$. Denote by $\overline{M}$ the clique partition of $G + K_q$ obtained with all $C \times \{i\}$ such that $C \in P_1$ and $i \leq q$ and with the vertical cliques of $\overline{L}$. Since $|\overline{M}| = |\overline{L}|$, the clique partition $\overline{M}$ is also minimum; moreover, every $C \in P_1$ is of cardinality $\geq q$ (otherwise, a C with less than q elements could be associated in $G + K_q$ with $|C| < q$ vertical cliques, which gives a better clique partition and contradicts the minimality of $\overline{L}$).

Now if P denotes the clique partition of G obtained by adding to the family P_1 as many singletons as needed, we have:

$$\theta_q(G) \leq B_q(P) = |\overline{M}| = \theta(G + K_q). \tag{3}$$

From (2) and (3), we get the required equality: $\theta_q(G) = \theta(G + K_q)$.

Lemma 3. *A graph G satisfies $\alpha_2(G) = \theta_2(G)$ if and only if $\alpha(G + K_2) = \theta(G + K_2)$.*

This follows the from lemmas 1 and 2.

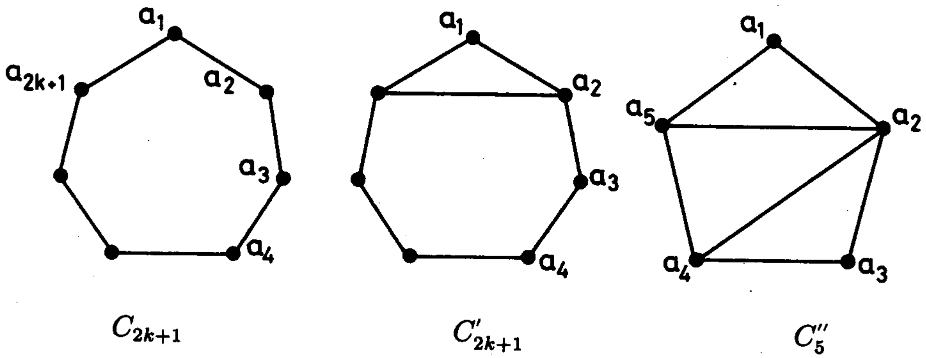


Figure 2.

Now, denote by C_{2k+1} the cycle of length $2k+1$; denote by C'_{2k+1} the graph obtained from C_{2k+1} by connecting a pair of vertices at distance 2 by an additional edge that we shall call a "short chord"; denote by C''_5 the cycle of length 5 with two non-crossing short chords (see Figure 2). Clearly, none of these are parity graphs, because the vertices a_1 and a_3 are connected by a minimal chain which is odd and by a minimal chain which is even. Burlet and Uhry [3] proved that a *graph is a parity graph if and only if* it *contains no induced C_{2k+1} with $k \geq 2$, no induced C'_{2k+1} with $k \geq 2$, and no induced C''_5*. Ravindra and Parthasarathy [10] have proved that *if G contains no induced C_{2k+1} with $k \geq 2$, no induced C'_{2k+1} with $k \geq 2$, and no induced C''_5, then $G + K_2$ is perfect* (For a more complete proof see de Werra and Hertz [5]).

Combining these results, we obtain:

Theorem 2. *Every parity graph is 2-perfect; furthermore the graph $G+K_2$ is perfect if and only if G is a parity graph.*

Proof. Let G be a parity graph. By the Ravindra-Parthasarathy theorem, the graph $G+K_2$ is perfect, and for all $A \subset V(G)$, $G_A + K_2$ is an induced subgraph of $G + K_2$. Hence:

$$\alpha_2(G_A) = \alpha(G_A + K_2) = \theta(G_A + K_2) = \theta_2(G_A).$$

Thus, G is 2-perfect.

Now, if G is not a parity graph, the Burlet–Uhry theorem states that either G contains the first graph of Figure 2, and then $G + K_2$ contains an induced C_{2k+3} : $a_1 1$, $a_2 1$, $a_3 1$, $a_3 2$, $a_4 2$, $a_5 2$, ..., $a_1 2$, $a_1 1$; or G contains the same induced C_{2k+3} as above; or G contains the third graph of Figure 2, and then $G + K_2$ contains an induced C_7 consisting of: $a_1 1$, $a_2 1$, $a_3 1$, $a_3 2$, $a_4 2$, $a_5 2$, $a_1 2$, $a_1 1$.

Thus, in all cases, the graph $G + K_2$ cannot be perfect. ∎

Theorem 3. *Let G be a graph whose maximum cliques are triangles and whose odd cycles of length ≥ 5 admit a chord. Then $\alpha_2(G) = \theta_2(G)$ iff the hypergraph G^3 on $V(G)$ whose edges are the triangles of G has the König property: the maximum number of triangles which are pairwise vertex-disjoint is equal to the least number of vertices which represent all the triangles.*

Proof. Assume that the matching number $\nu(G^3)$ is equal to the transversal number $\tau(G^3)$ (the "König property", see [1]). Denote by H^0 the hypergraph of the odd cycles in G. Clearly,

$$\alpha_2(G) = n - \tau(H^0) = n - \tau(G^3)$$

(because every odd cycle has a chord). A 2–optimal clique partition M of G contains exactly $\nu(G^3)$ triangles; so by the König property,

$$n - \tau(G^3) = n - \nu(G^3) = B_2(M) = \theta_2(G).$$

Thus, the König property for G^3 implies that $\alpha_2(G) = \theta_2(G)$.

Similarly, $\alpha(G) = \theta_2(G)$ implies that $\nu(G^3) = \tau(G^3)$. ∎

Corollary. *Let G be a graph such that*

(i) the maximum cliques are triangles,

(ii) every odd cycle of length ≥ 5 *has a chord,*

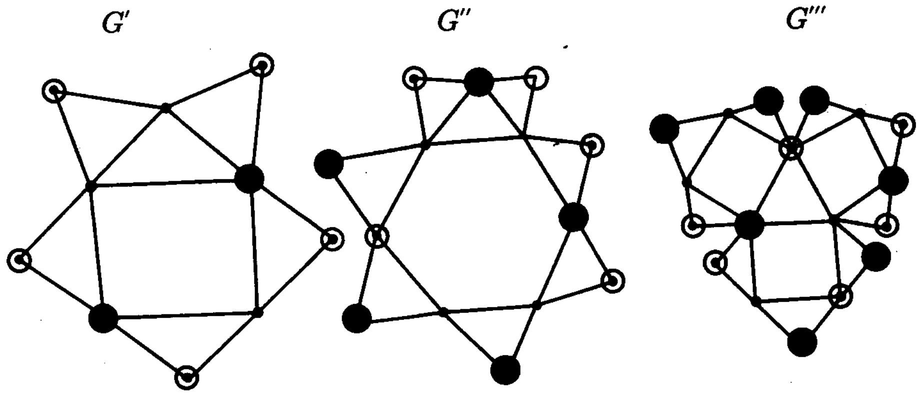
(iii) the Hajós graph (Figure 1) is not a partial subgraph of G,

(iv) the intersection graph of the triangles is perfect.

Then G is 2-perfect.

Proof. (i) and (iii) imply that G^3 has the Helly property: a family of pairwise intersecting triangles have a non empty intersection (see [1]). Since the intersection graph $L(G^3)$ is perfect, this implies that G^3 is normal and that G^3 has the König property ([1], Proposition 1, Chap. 1). ■

Open Problem. The smallest minimal graph which is perfect but not 2-perfect is the Hajós graph; we can also mention the graphs G', G'', G''' of Figure 3.



$\alpha_2(G') = 7$ $\theta_2(G') = 8$

$\alpha_2(G'') = 10$ $\theta_2(G'') = 11$

$\alpha_2(G''') = 13$ $\theta_2(G''') = 14$

Figure 3.

All the perfect graphs which are minimal non-2-perfect seem to contain an odd cycle which induces a perfect graph (but does not admit crossing chords), together with a "wing" attached to each of its sides; we call *wing* of a cycle μ a triangle abc obtained from μ by adding a vertex $c \notin \mu$ to two consecutive vertices a, b of μ, and by adding the edges ac and bc. If a perfect graph is not 2-perfect, does it necessarily contain this kind of configuration?

Remark. The graphs G', G'', G''' on Figure 3 are 3-perfect but not 2-perfect. On the other hand, a 2-perfect graph need not be 3-perfect. For instance, the graph G consisting of a clique $\{a, b, c, d, e, \}$ together with five cliques $\{a, b, a_1, a_2\}$, $\{b, c, b_1, b_2\}$, $\{c, d, c_1, c_2\}$, $\{d, e, d_1, d_2\}$ and $\{e, a, e_1, e_2\}$, is 2-perfect, but not 3-perfect since $\alpha_3(G) = 12$ and $\theta_3(G) = 13$.

Similarly, with one clique K_{2k-1} and $2k - 1$ cliques K_{k+1}, one can produce a graph which is k-perfect but not $(k + 1)$-perfect. The graphs which are q-perfect for all integers q will be discussed on another occasion.

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