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Rado's Theorem for Finite Fields

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1. Introduction

Over half a century ago, R. Rado was interested in finding the right setting for results like Schur's Theorem and van der Waerden's Theorem on arithmetic progressions. Schur's Theorem [15] says that when the set $\mathbb{N}$ of positive integers is divided into finitely many classes, one of these classes contains x, y, and $x+y$ while van der Waerden's Theorem [16] says that one of these classes contains arbitrarily long arithmetic progressions. He found this setting by considering solutions of systems of homogeneous linear equations. In [13] Rado proved his famous theorem characterizing the matrices of those systems of homogeneous linear equations with integral coefficients which are partition regular over the set $\mathbb{N}$ of positive integers. This theorem revolves around what is known as the "columns condition" which we will state in some generality.

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Definition 1.1. *Let R be an integral domain, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix with entries from R. Then C satisfies the columns condition over R if and only if the columns of C can be ordered $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_v$ so that there exist $m \in \mathbb{N}$ and $1 \leq k_1 < k_2 < \ldots < k_m = v$ such that*

(1) $\Sigma_{i=1}^{k_1} \vec{c}_i = \vec{0}$ and

(2) for $t \in \{2, 3, \ldots, m\}$ there exist $\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{k_{t-1},t}$ in R and $d_t \in R\backslash\{0\}$ with $d_t \cdot \Sigma_{i=k_{t-1}+1}^{k_t} \vec{c}_i = \Sigma_{i=1}^{k_{t-1}} \alpha_{i,t} \cdot \vec{c}_i$.

Definition 1.2. *Let R be a commutative ring, let $u, v \in \mathbb{N}$, let C be a $u \times v$ matrix over R, and let M be a module over R. Then C is partition regular over some subset B of M if and only if whenever B is divided into finitely many classes one of these classes contains $x_1, x_2, \ldots, x_v$ with $C\vec{x} = 0$.*

With this terminology, Rado's original theorem is:

Theorem 1.3. (Rado [13]). *Let $u, v \in \mathbb{N}$ and let C be a $u \times v$ matrix over $\mathbb{Z}$. Then C is partition regular over $\mathbb{N}$ if and only if C satisfies the columns condition over $\mathbb{Z}$.*

As an illustration consider Schur's Theorem, van der Waerden's Theorem and the finite version of the Finite Sum Theorem. (As we will see it is easily derivable from Rado's Theorem. However, it was independently discovered by Folkman and by Sanders. See [6] or [11].) Schur's Theorem can be phrased as saying that whenever $\mathbb{N}$ is finitely colored one can obtain a monochrome solution to the equation $x_1 + x_2 = x_3$. The coefficient matrix is (1,1,-1); rearranging its columns we get (1,-1,1). Then with $k_1 = 2$, $k_2 = 3$, $d_2 = 1$, $\alpha_{1,2} = 1$, and $\alpha_{2,2} = 0$ we see that the columns condition is satisfied. (In fact, with only one equation, the columns condition is equivalent to the assertion that some nonempty subset of the coefficients sums to 0.)

Now consider a strengthened version of van der Waerden's Theorem. Given a finite coloring of $\mathbb{N}$ we want to find, say, a length 5 arithmetic progression a, $a+d$, $a+2d$, $a+3d$, $a+4d$ which, along with its increment d, is monochrome. Let $x_1 = d$, $x_2 = a$, $x_3 = a+d$, $x_4 = a+2d$, $x_5 = a+3d$, and $x_6 = a+4d$. Then the coefficient matrix is

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 3 & 1 & 0 & 0 & -1 & 0 \\ 4 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Reversing the columns we have

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 & 1 & 3 \\ -1 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

With $k_1 = 5$, $k_2 = 6$, $d_2 = 1$, $\alpha_{1,2} = -4$, $\alpha_{2,2} = -3$, $\alpha_{3,2} = -2$, $\alpha_{4,2} = -1$, and $\alpha_{5,2} = 0$, we see that the columns condition is satisfied.

Consider finally the length three version of the Finite Sums Theorem. This says that whenever $\mathbb{N}$ is finitely partitioned, one class contains some $y_1, y_2, y_1 + y_2, y_3, y_3 + y_1, y_3 + y_2$, and $y_3 + y_1 + y_2$.

As a system of linear equations this may be formulated as

$$\begin{pmatrix} 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ x_7 \end{pmatrix} = \vec{0}$$

Observe that the sum of columns 1,4,5,6 is zero. Moreover the sum of columns 2,7 is a linear combination of columns 4,5 and finally column 3 is a combination of columns 4,6,7.

In 1939, Rado proved a significant strengthening of his original result.

Theorem 1.4. (Rado [14,Theorem VII]). *Let R be any subring of the complex numbers, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix over R. Then C is partition regular over $R \setminus \{0\}$ if and only if C satisfies the columns condition over R.*

In this paper, we establish that a finitistic version of Rado's Theorem is valid for vector spaces over any finite field. We derive this result in two entirely different ways which we present nearly independently. Specifically we obtain as Corollary 2.5 and Theorem 3.4 the following

Theorem. *Let F be a finite field, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix over F. The following statements are equivalent.*

(a) For each $r \in \mathbb{N}$ there is some $m \in \mathbb{N}$ so that whenever $n \geq m$ and V is an n-dimensional vector space over F, and $V \setminus \{0\}$ is r-colored, there exist monochrome $x_1, x_2, \ldots, x_v \in V \setminus \{0\}$ with $C\vec{x} = \vec{0}$.

(b) C satisfies the columns condition over F.

We present in Section 2 a proof using standard (though in places high powered) combinatorial methods. In Section 3, we utilize the notion of central sets, whose definition depends on the algebraic structure of the Stone-Čech compactification of a discrete semigroup. In Section 4 we present several consequences of the main results.

2. A classical derivation

We shall make use of the Ramsey Theorem of Graham, Leeb and Rothschild for spaces over a finite field.

Theorem 2.1. (Graham, Leeb, and Rothschild [10]). *Let F be a finite field and let $m, r \in \mathbb{N}$ be given. If n is large enough then for every r-coloring of the projective points in F^n (one-dimensional subspaces), there exists an m-dimensional space which is monochromatic.*

As a second preliminary we need the following fact whose proof is based on the corresponding proof for $F = \mathbb{Z}_p$ in [5].

Lemma 2.2. *Let F be a finite field and let V be a vector space over F. Let $u, v \in \mathbb{N}$ and let C be a $u \times v$ matrix over F. There is a partition of $V \setminus \{0\}$ into $|F| - 1$ cells so that if there exist $x_1, x_2, \ldots, x_v$ in the same cell of the partition with $C\vec{x} = \vec{0}$, then C satisfies the columns condition over F.*

Proof. We may presume that $V = \oplus F$. We color the nonzero elements of V according to the value of their first nonzero coordinate. That is, for $x \in V \setminus \{0\}$ let $\delta(x) = \min\{\gamma : x(\gamma) \neq 0\}$ and for $\alpha \in F \setminus \{0\}$, let $A_\alpha = \{x \in V \setminus \{0\} : x(\delta(x)) = \alpha\}$. Then $\{A_\alpha : \alpha \in F \setminus \{0\}\}$ is a partition of $V \setminus \{0\}$ into $|F| - 1$ classes. Now let C be a $u \times v$ matrix over F and assume we have $\mu \in F \setminus \{0\}$ and $x_1, x_2, \ldots, x_v$ in A_μ with $C\vec{x} = \vec{0}$. By rearranging the $x_i's$ and the columns of C we may presume $\delta(x_1) \leq \delta(x_2) \leq \ldots \leq \delta(x_v)$.

Choose $m \in \mathbb{N}$ and $k_1, k_2 \ldots, k_m$ in $\mathbb{N}$ with $1 \leq k_1 < k_2 < \ldots < k_m = v$ and $\delta(x_1) = \delta(x_2) = \ldots = \delta(x_{k_1})$ and for $t \in \{1, 2, \ldots, m-1\}$, $\delta(x_{k_t+1}) = \delta(x_{k_t+2}) = \ldots = \delta(x_{k_{t+1}})$ and $\delta(x_{k_t}) < \delta(x_{k_{t+1}})$. For $t \in \{2, 3, \ldots, m\}$, let $d_t = -\mu$ and for $i \in \{1, 2, \ldots, k_{t-1}\}$ let $\alpha_{i,t} = x_i(\delta(x_{k_t}))$.

By assumption class A_μ contains a solution of $C\vec{x} = 0$ with $x_i \in V = \oplus F$. Therefore $C\vec{x} = \vec{0}$ as matrix equation. Now focus attention to the column $\vec{x}(\delta(x_1))$ with index $\delta(x_1)$ and obtain $C\vec{x}(\delta(x_1)) = 0$. As $x_i(\delta(x_1)) = 0$ for $i > k_1$, obtain $\Sigma_{j=1}^{k_1} \vec{c}_j x_j(\delta(x_1)) = \vec{0}$, where $\vec{c}_j$ is the j'th row of C. Finally $\mu\Sigma_{j=1}^{k_1} \vec{c}_j = \vec{0}$ and as $\mu \neq 0$ we have $\Sigma_{j=1}^{k_1} \vec{c}_j = \vec{0}$.

Now let $t \in \{2, 3, \ldots, m\}$ and focus attention to the column $\vec{x}(\delta(x_t))$ of X with index $\delta(x_t)$ and obtain $C\vec{x}(\delta(x_t)) = \vec{0}$. Again $x_i(\delta(x_t)) = 0$ for $i > k_t$ yielding $\Sigma_{j=1}^{k_t} \vec{c}_j x_j(\delta(x_t)) = \vec{0}$. By splitting the summation and using the definition of the partition

$$\Sigma_{j=1}^{k_{t-1}} \vec{c}_j x_j(\delta(x_{k_t})) + \mu\Sigma_{j=k_{t-1}+1}^{k_t} \vec{c}_j = \vec{0}$$

as required.

As a final preliminary we have:

Lemma 2.3. *Let F be a finite field, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix over F which satisfies the columns condition over F with m classes. Let V be an $\aleph_0$-dimensional vector space over F and let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m \in V$ be linearly independent. Let $F(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m) = \{\vec{x}_i + \Sigma_{j=i+1}^{m} a_{i,j}\vec{x}_j : i \in \{1, 2, \ldots, m\}$ and each $a_{i,j} \in F\}$. Then there exist vectors $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m$ in $F(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m)$ with $C(\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m)^T = \vec{0}$.*

Proof. The proof in [5] for $F = \mathbb{Z}_p$ carries over. ∎

The infinitary version of our result can now be presented.

Theorem 2.4. *Let F be a finite field and let V be an $\aleph_0$-dimensional vector space over F. Let $u, v \in N$ and let C be a $u \times v$ matrix over F. The following statements are equivalent.*

(a) *C is partition regular over $V \setminus \{0\}$.*

(b) *C satisfies the columns condition over F.*

Proof. That $(a) \Rightarrow (b)$ follows immediately from Lemma 2.2.

In order to prove that $(b) \Rightarrow (a)$, let C satisfy the columns condition over F and let $\Delta : V \to \{1, 2, \ldots, r\}$ be a coloring. In order to define a coloring of the 1-dimensional subspaces in V, choose in each such subspace S an element s with leading coefficient 1 in the coordinate representation and let $\Delta'(S) = \Delta(s)$. By Theorem 2.1 find a monochromatic v-dimensional subspace. Then the representatives chosen above form a set $F(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_v)$, which is monochromatic. Thus by Lemma 2.3 $C\vec{y} = \vec{0}$ has a monochromatic solution for Δ. ∎

Corollary 2.5. *Let F be a finite field, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix over F. The following statements are equivalent.*

(a) For each $r \in \mathbb{N}$ there is some $m \in \mathbb{N}$ so that whenever $n \geq m$, V is an n-dimensional vector space over F, and $V \setminus \{0\}$ is r-colored, there exist monochrome $x_1, x_2, \ldots, x_v$ with $C\vec{x} = \vec{0}$.

(b) C satisfies the columns condition over F.

Proof. That $(b) \Rightarrow (a)$ is established from Theorem 2.4 using a standard compactness argument. (See for example the proof of Theorem 3.4.)

To see that $(a) \Rightarrow (b)$ let V_n be an n-dimensional vector space over F (with $V_n \subseteq V_{n+1}$) and let $V = \bigcup_{n=1}^{\infty} V_n$. Then C is partition regular over $V \setminus \{0\}$ so Theorem 2.4 applies. ∎

Corollary 2.6. *Let F be a finite field, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix over F. Let V be an $\aleph_0$-dimensional vector space over F. Then C is partition regular over V if and only if there exist linearly independent vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_v$ such that $F(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_v)$ contains a solution of $C\vec{y} = \vec{0}$ (and then for any set $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_v$ of linearly independent vectors, $F(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_v)$ will contain a solution of $C\vec{y} = \vec{0}$).*

Proof. We have seen that if C satisfies the columns condition, then every set $F(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_v)$ contains a solution of $C\vec{x} = \vec{0}$. On the other hand, if $F(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_v)$ contains a solution of $C\vec{x} = \vec{0}$, it is a standard calculation to establish that C satisfies the columns condition over F. (See [11] for example.)

3. A derivation via central sets

Our starting point is the following generalization of [8, Proposition 8.21] which was proved in [3, Corollary 2.10]. A subset A of a semigroup $(S, +)$ is central if and only if A is a member of some idempotent in the smallest ideal of $(\beta S, +)$, where βS is the Stone-Čech compactification of the discrete space S and $+$ is the associative left continuous extension of the operation on S which has S contained in the topological center. As far as this paper is concerned, we only need to know that if $(G, +)$ is an infinite group and $G \setminus \{0\}$ is partitioned into finitely many classes, then one of these classes is central in G. (See [2] and [12] for more information about $(\beta S, +)$ and central sets.)

Theorem 3.1. *Let $(S, +)$ be a countable commutative semigroup, let A be central in S, let $\ell \in \mathbb{N}$, and for each $i \in \{1, 2, \ldots, \ell\}$ let $\langle y_{i,m}\rangle_{m=1}^{\infty}$ be a sequence in S. There exist a sequence $\langle a_t\rangle_{t=1}^{\infty}$ in S, a sequence $\langle H_t\rangle_{t=1}^{\infty}$ of pairwise disjoint finite nonempty subsets of $\mathbb{N}$, and for each $i \in \{0, 1, \ldots, \ell\}$, a sequence $\langle z(t, i)\rangle_{t=1}^{\infty}$ in S such that*

(1) for $t \in \mathbb{N}$, $z(t, 0) = a_t$

(2) for $t \in \mathbb{N}$ and $i \in \{1, 2, \ldots, \ell\}$, $z(t, i) = a_t + \Sigma_{m \in H_t} y_{i,m}$, and

(3) whenever F is a finite nonempty subset of $\mathbb{N}$ and $f : F \to \{0, 1, 2, \ldots, \ell\}$ one has $\Sigma_{t \in F}\, z(t, f(t)) \in A$.

The next result utilizes the idea of (m, p, c)-sets from [4] and [5]. It will be convenient in this lemma to distinguish between an infinite dimensional vector space over F (whose representation we do not care about) and the set $\oplus_{i=1}^{\infty} F$ of all sequences in F with only finitely many non-zero terms.

Lemma 3.2. *Let F be a finite or countably infinite field and let V be an $\aleph_0$-dimensional vector space over F. Let L be a finite nonempty subset of $\oplus_{i=1}^{\infty} F$ such that for each $\lambda \in L$ some $\lambda_i \neq 0$ and if $j = \min\{i : \lambda_i \neq 0\}$, then $\lambda_j = 1$. Let $m = \max\{i :$ there exist $\lambda \in L$ with $\lambda_i \neq 0\}$ and let G be a finite partition of $V \setminus \{0\}$. Then there exist $A \in G$ and for each $i \in \{1, 2, \ldots, m\}$, a sequence $\langle x_{i,n}\rangle_{n=1}^{\infty}$ in V such that $\{\Sigma_{i=1}^{m} (\lambda_i \cdot \Sigma_{n \in H}\, x_{i,n}) :$ $\lambda \in L$ and H is a finite nonempty subset of $\mathbb{N}\} \subseteq A$.*

Proof. Pick $A \in G$ which is central in $(V, +)$. For $j \in \{1, 2, \ldots, m\}$, let $L_j = \{\lambda \in L : j = \min\{i : \lambda_i \neq 0\}\}$. We show by downward induction on $r \in \{1, 2, \ldots, m\}$ that there exist sequences $\langle x_{i,n}\rangle_{n=1}^{\infty}$ in V for each $i \in \{r, r+1, \ldots, m\}$ such that $\{\Sigma_{i=r}^{m} (\lambda_i \cdot \Sigma_{n \in H}\, x_{i,n}) : \lambda \in \cup_{i=r}^{m} L_i$ and H is a finite nonempty subset of $\mathbb{N}\} \subseteq A$.

First assume $r = m$ and pick a sequence $\langle x_{m,n}\rangle_{n=1}^{\infty}$ with $\Sigma_{n \in H}\, x_{m,n} \in A$ whenever H is a finite nonempty subset of $\mathbb{N}$. (For example take $x_{m,n} = a_n$ in Theorem 3.1 and in conclusion (3) let f be constantly 0.) Now L_m, if nonempty, consists of one λ with $\lambda_m = 1$ and $\lambda_i = 0$ otherwise. Thus $\{\lambda_m \cdot \Sigma_{n \in H}\, x_{m,n} : \lambda \in L_m$ and H is a finite nonempty subset of $\mathbb{N}\} \subseteq A$.

Now assume $r < m$ and pick for each $i \in \{r+1,\ r+2, \ldots, m\}$ a sequence $\langle x'_{i,n}\rangle_{n=1}^{\infty}$ as guaranteed by the induction hypothesis. For each $\lambda \in L_r$ and each $n \in \mathbb{N}$, let $y_{\lambda,n} = \Sigma_{i=r+1}^{m} \lambda_i x'_{i,n}$. (We may presume

$L_r \neq \emptyset$, since we may add λ with $\lambda_r = 1$ and $\lambda_i = 0$ for $i \neq r$.) We apply Theorem 3.1 with $\ell = |L_r|$. Pick sequences $\langle a_t \rangle_{t=1}^{\infty}$, $\langle H_t \rangle_{t=1}^{\infty}$, and for each $\lambda \in L_r$, a sequence $\langle z(t,\lambda) \rangle_{t=1}^{\infty}$ as guaranteed by Theorem 3.1. Then taking the functions f in conclusion (3) to be constant we have for each finite nonempty subset K of $\mathbb{N}$ and each $\lambda \in L_r$ that $\Sigma_{t \in K} (a_t + \Sigma_{n \in H_t} y_{\lambda,n}) \in A$. For $t \in \mathbb{N}$, let $x_{r,t} = a_t$ and for $i \in \{r+1, r+2, \ldots, m\}$, let $x_{i,t} = \Sigma_{n \in H_t} x'_{i,n}$. Now let $\lambda \in \bigcup_{i=r}^{m} L_i$ and let K be a finite nonempty subset of $\mathbb{N}$. Assume first $\lambda \in \bigcup_{i=r+1}^{m} L_i$ and let $G = \bigcup_{t \in K} H_t$. Then $\lambda_r = 0$ so $\Sigma_{i=r}^{m} (\lambda_i \cdot \Sigma_{t \in K} x_{i,t}) = \Sigma_{i=r+1}^{m} (\lambda_i \Sigma_{t \in K} \Sigma_{n \in H_t} x'_{i,n}) = \Sigma_{i=r+1}^{m} (\lambda_i \Sigma_{n \in G} x'_{i,n}) \in A$ by the induction hypotheses. Now assume $\lambda \in L_r$, so $\lambda_r = 1$. Then $\Sigma_{i=r}^{m} (\lambda_i \cdot \Sigma_{t \in K} x_{i,t}) = \Sigma_{t \in K} a_t + \Sigma_{i=r+1}^{m} (\lambda_i \cdot \Sigma_{t \in K} \Sigma_{n \in H_t} x'_{i,n}) = \Sigma_{t \in K} a_t + \Sigma_{t \in K} \Sigma_{n \in H_t} \Sigma_{i=r+1}^{m} \lambda_i x'_{i,n} = \Sigma_{t \in K} (a_t + \Sigma_{n \in H_t} y_{\lambda,n}) \in A$. ■

We can now prove the sufficiency of Rado's Theorem for any $\aleph_0$-dimensional vector space over a countable field.

Theorem 3.3. *Let F be a finite or countably infinite field and let V be an $\aleph_0$-dimensional vector space over f, let $u, v \in \mathbb{N}$ and let C be a $u \times v$ matrix over F which satisfies the columns condition over F. Then C is partition regular over $V \setminus \{0\}$.*

Proof. Let G be a finite partition of $V \setminus \{0\}$. Pick $k_1, k_2, \ldots, k_m$ and for $t \in \{2, 3, \ldots, m\}$ pick $\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{k_{t-1},t}$ and d_t as guaranteed by the columns condition for C (whose columns we presume have been reordered, if necessary). Since each $d_t \in F \setminus \{0\}$, we may presume each $d_t = 1$ (and hence $\Sigma_{i=k_{t-1}+1}^{k_t} \vec{c}_i = \Sigma_{i=1}^{k_{t-1}} \alpha_{i,t} \vec{c}_i$ for each $t \in \{2, \ldots, m\}$). We define $L = \{\lambda_1, \lambda_2, \ldots, \lambda_v\}$ as follows (using functional notation for the coordinates of λ_i). Given $i \in \{1, 2, \ldots, v\}$ and $j \in \mathbb{N}$, let

$$\lambda_i(j) = \begin{cases} 1 & \text{if} \quad j = 1 \quad \text{and} \quad i \leq k_1 \\ 1 & \text{if} \quad 1 < j \leq m \quad \text{and} \quad k_{j-1} < i \leq k_j \\ -\alpha_{i,j} & \text{if} \quad 1 < j \leq m \quad \text{and} \quad i \leq k_{j-1} \\ 0 & \text{if} \quad j \leq m \quad \text{and} \quad i > k_j \\ 0 & \text{if} \quad j > m \end{cases}$$

Observe that for each $j \leq m$ we have $\Sigma_{i=1}^{v} \lambda_i(j) \cdot \vec{c}_i = \vec{0}$.

Now L satisfies the hypotheses of Lemma 3.2 so pick $A \in G$ and for each $j \in \{1, 2, \ldots, m\}$ pick a sequence $\langle y_{j,n} \rangle_{n=1}^{\infty}$ in V such that $\{\Sigma_{j=1}^{m} (\lambda_i(j) \cdot \Sigma_{n \in H} \lambda_{j,n}) : \lambda_i \in L$ and H is a finite nonempty subset of $\mathbb{N}\} \subseteq A$. In particular, letting $H = \{1\}$ we have for each $i \in \{1, 2, \ldots, v\}$ that $\Sigma_{j=1}^{m} \lambda_i(j) \cdot$

$y_{j,1} \in A$. Let for each $i \in \{1,2,\ldots,v\}$, $x_i = \Sigma_{j=1}^m \lambda_i(j) \cdot y_{j,1}$. Then each $x_i \in A$ and $\vec{x} = \Sigma_{i=1}^v \vec{c}_i x_i = \Sigma_{i=1}^v \vec{c}_i \cdot (\Sigma_{j=1}^m \lambda_i(j) \cdot y_{j,1}) = \Sigma_{j=1}^m (y_{j,1}) \cdot \Sigma_{i=1}^v \lambda_i(j) \cdot \vec{c}_i = \Sigma_{j=1}^m y_{j,i} \cdot \vec{0} = \vec{0}$. ∎

Theorem 3.4. *Let F be a finite field, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix over F. The following statements are equivalent.*

(a) For each $r \in \mathbb{N}$ there is some $m \in \mathbb{N}$ so that whenever $n \geq m$, V is an n-dimensional vector space over F, and $V \setminus \{0\}$ is r-colored, there exist monochrome $x_1, x_2, \ldots, x_v$ with $C\vec{x} = \vec{0}$.

(b) C satisfies the columns condition over F.

Proof. (a)⇒(b). Let $r = |F| - 1$, pick m as guaranteed for r, and apply Lemma 2.2.

(b)⇒(a). For each n let V_n be the n-dimensional vector space over F and assume $V_n \subseteq V_{n+1}$. We proceed by a standard compactness argument.

Let $r \in \mathbb{N}$ and suppose the conclusion fails. For cofinally many n's, and hence for all n, we may pick a function $\varphi_n : V_n \setminus \{0\} \to \{0, 1, \ldots, v-1\}$ so that for each $i < r$, $\varphi_n^{-1}[\{i\}]$ does not contain $x_1, x_2, \ldots, x_v$ with $C\vec{x} = \vec{0}$. Choose an infinite subset A_1 of $\mathbb{N}$ so that for $n, m \in A_1$ one has $\varphi_n(V_1 \setminus \{0\}) = \varphi_m(V_1\{0\})$. Inductively, given A_{t-1}, choose an infinite subset A_t of A_{t-1} so that min $A_t \geq t$ and for $n, m \in A_t$ one has $\varphi_n(V_t \setminus \{0\}) = \varphi_m(V_t \setminus \{0\})$. For each t pick $n(t) \in A_t$. Let $V = \cup_{n=1}^\infty V_n$. Then V is an $\aleph_0$-dimensional vector space over F. For $x \in V \setminus \{0\}$ pick the first t such that $x \in V_t$ and define $\varphi(x) = \varphi_{n(t)}(x)$. By Theorem 3.3 pick $i < r$ and $x_1, x_2, \ldots, x_v$ in $\varphi^{-1}(i)$ with $C\vec{x} = \vec{0}$. Pick t with $\{x_1, x_2, \ldots, x_v\} \subseteq V_t$. Then for each $j \in \{1, 2, \ldots, v\}$ one has $\varphi_{n(t)}(x_j) = i$. (For j pick the least s such that $x_j \in V_s$. Then $n(s), n(t) \in A_s$ so $\varphi_{n(t)}(x_j) = \varphi_{n(s)}(x_j) = \varphi(x_j) = i$.) Since $\{x_1, x_2, \ldots, x_v\} \subseteq V_t \subseteq V_{n(t)}$, this is a contradiction. ∎

4. Some consequences of the main result

When Schür proved his famous combinatorial result [15], he was interested in solving the equation $x^n + y^n = z^n$ in $\mathbb{Z}_p$. He proved that for each n there is an m so that, if p is a prime bigger than m, the equation $x^n + y^n = z^n$ is solvable in $\mathbb{Z}_p$. Equivalently, the equation $x+y=z$ is solvable in $\{a^n : a \in \mathbb{Z}_p\setminus\{0\}\}$. Compare the following corollary.

Corollary 4.1. *Let F be a finite field, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix which satisfies the columns condition over F. Then for each n and r in $\mathbb{N}$ there exists $m \in \mathbb{N}$ so that whenever K is a field extension of F and $[K : F] \geq m$ and $\{a^n : a \in F\}$ is r-colored, then there exist monochrome $x_1, x_2, \ldots, x_v$ with $C\vec{x} = \vec{0}$.*

Proof. Let $r' = n \cdot r$ and pick $m \in \mathbb{N}$ as guaranteed by the main result for r'. Let K be a field extension of F with $[K : F] \geq m$, let $\Gamma = \{x^n : x \in K \setminus \{0\}\}$ and let $\Gamma = \cup_{i=1}^{r} B_i$. Now Γ is a multiplicative subgroup of $K \setminus \{0\}$ with index at most n. Pick $t \leq n$ and $z_1, z_2, \ldots, z_t$ in K so that $\Gamma_{z_1}, \Gamma_{z_2}, \ldots, \Gamma_{z_t}$ are the cosets of Γ. For $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, t\}$, let $D_{i,j} = \{x \cdot z_j : x \in B_i\}$. Then $K\setminus\{0\}$ is partitioned into $r \cdot t \leq r'$ cells so pick $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, t\}$ and $y_1, y_2, \ldots, y_v$ in $D_{i,j}$ with $C\vec{y} = \vec{0}$. Let $x_1 = y_1 z_j^{-1}, x_2 = y_2 z_j^{-1}, \ldots, x_v = y_v z_j^{-1}$. Then $\{x_1, x_2, \ldots, x_v\} \subseteq B_i$ and $C\vec{x} = C\vec{y} z_j^{-1} = \vec{0} z_j^{-1} = \vec{0}$. ∎

The following result tells us that a solution can always be obtained in one "good" color.

Corollary 4.2. *Let F be a finite field and let V be an $\aleph_0$-dimensional vector space over F and let $\Delta : V\setminus\{0\} \to \{1, 2, \ldots, r\}$. Then one of the colors i satisfies: for every partition regular matrix C there exists a solution of $C\vec{x} = \vec{0}$ within $\Delta^{-1}(i)$. That is, one color class is universal.*

Proof. In terms of Section 2 one may take larger and larger sets of the form $F(x_1, x_2, \ldots, x_v)$, $v < \aleph_0$, and choose a color occurring cofinally.
In terms of Section 3, one simply chooses i so that $\Delta^{-1}(i)$ is central. ∎

The above corollary is interesting for some different reasons. In the first place it shows that Rado's universal sets (see [4]) play the same role for arbitrary vector spaces as for integers. One also sees that the diagonal sum of all partition regular matrices is partition regular again. From these observations we have:

Corollary 4.3. *Let F be a finite field and let V be an $\aleph_0$-dimensional vector space over F. Let $V \setminus \{0\}$ be partitioned into finitely many classes. Then one of these classes contains solutions to all partition regular matrices.*

Further if $\langle C_i \rangle_{i=1}^{\infty}$ is an enumeration of these matrices there exist solutions $\vec{x}(i)$ to $C_i\vec{x}(i) = \vec{0}$ so that whenever H is a finite non-empty subset of $\mathbb{N}$ and $f(i)$ chooses a coordinate of $\vec{x}(i)$ for each $i \in H$, one has $\Sigma_{i \in H}\, \vec{x}(i)_{f(i)}$ is in this same cell.

Proof. Again choose a central class. See [7]. ∎

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