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On Superspherical Graphs

S. L. BEZRUKOV and A. SALI[†‡]

ABSTRACT

In this paper we consider two conjectures of H. M. Mulder [2] and give partial solutions for them. First we prove that a triangle free superspherical graph is interval regular provided it satisfies an additional (not too strong) condition. Furthermore, we show that a spherical interval regular graph is interval monotone.

1.Introduction

Graphs that are well-structured or highly symmetrical are studied extensively. In particular hypercubes and graphs that are close to hypercubes draw much attention. In his book [2] H.M. Mulder put forward the following conjecture (see the definitions below).

Conjecture 1.1. *An interval regular graph is interval monotone.*

Furthermore, he proposed the following.

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Conjecture 1.2. *A triangle free superspherical graph is interval regular.*

In the present paper we prove two related theorems. In order to formulate the results we need some definitions.

Definition 1.3. *Let $G = (V, E)$ be a finite graph and let $u, v \in V$. The* interval *$I(u, v)$ is the subgraph of G induced by the set of all vertices lying on a shortest path between u and v. The* length *of the interval is the distance $d(u, v)$ of u and v.*

Let us denote the d-dimensional hypercube by $\mathcal{B}_d$. So $\mathcal{B}_d$ is the graph whose vertex set is $\{0, 1\}^d$ and two vertices are connected by an edge iff they differ in exactly one position.

Definition 1.4. *Let $G = (V, E)$ be a finite connected graph. G is called* interval regular *if for any $u, v \in V$ the subgraph induced by the set of edges between levels in the interval $I(u, v)$ is a hypercube $\mathcal{B}_d$ with $d = d(u, v)$.*

Note that this definition is different from that of [2], but the two are equivalent. Furthermore, $I(u, v)$ need not be isomorphic to $\mathcal{B}_d$ even in the case of an interval regular G. Let us denote the property that *the subgraph induced by the set of edges between levels in the interval $I(u, v)$ is a hypercube $\mathcal{B}_d$ with $d = d(u, v)$* by $I(u, v) \bowtie \mathcal{B}_d$. The next definition deals with convexity properties of intervals.

Definition 1.5. *Let $G = (V, E)$ be a finite graph and let $A \subset V$. We say that A is* convex *if for every $x, y \in A$ the vertices of $I(x, y)$ are contained in A. G is* interval monotone *if each interval of G is convex. G is said to have the* quadrangle property, *if for every interval $I(u, v)$ of G and every $x, y \in I(u, v)$ such that $d(x, u) = 1$ and $d(y, u) = 1$, there exists a $u \neq z \in I(u, v)$ such that $d(z, x) = d(z, y) = 1$.*

One more definition is needed.

Definition 1.6. *Let $G = (V, E)$ be a finite graph and let $u, v \in V$ such that $d(u, v) = d$. Let $x, y \in I(u, v)$. We say that x and y are* diametrical *if $d(x, y) = d(u, v) = d$. G is called* spherical *if for every point in each of its intervals there exists at least one diametrical point in that particular interval. G is* superspherical *if spherical and for every point the diametrical pair is unique in every interval.*

If x and y are diametrical in some interval, then y is often called the *opposite* or *complement* of x in that interval. It is easy to see that $\mathcal{B}_d$ is an example of superspherical graphs. However, a superspherical graph need not be a hypercube as the following example shows. Let $\mathcal{B}_d(1,d)$ denote the graph whose vertex set is $\{0,1\}^d$ and two vertices are connected by an edge iff they differ in exactly one or d positions. Then $\mathcal{B}_{2t}(1,2t)$ is a superspherical graph. Figure 1 depicts $\mathcal{B}_4(1,4)$.

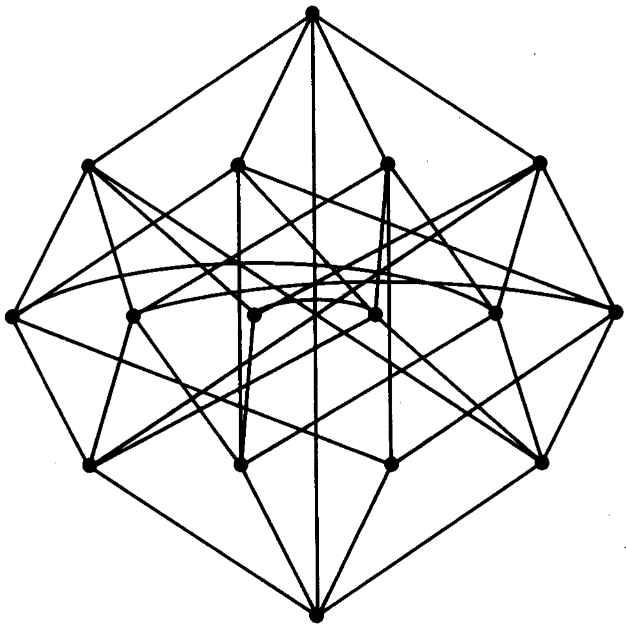

Fig. 1. $\mathcal{B}_4(1,4)$

Now we can formulate the main results.

Theorem 1.7. *If a triangle free superspherical graph G satisfies the quadrangle property, then it is interval regular.*

Theorem 1.8. *If an interval regular graph G is spherical, then it is interval monotone.*

As an immediate corollary we obtain the following.

Corollary 1.9. *For triangle free superspherical graphs the following three conditions are equivalent:*

(i) *G satisfies the quadrangle property*
(ii) *G is interval monotone*
(iii) *G is interval regular.* ■

We mention that Havel and Liebl proved that a bipartite superspherical graph is in fact isomorphic to a hypercube [1].

2. Proofs

The proof of Theorem 1.7 consists of several small steps. We use induction on d. Let α be a neighbour of v in $I(u,v)$. Let $\bar{\alpha}$ be its unique diametrical pair. Let level i consist of vertices in $I(u,v)$ of distance i from u. For $z \in I(u,v)$ let $l(z)$ denote the level of z. The first step is the case $d = 2$.

Step 1. *If x and y are such that $d(x,y) = 2$, then $I(x,y) \cong \mathcal{B}_2$.*

Proof. $I(x,y)$ consists of common neighbours of x and y besides x and y because $d(x,y) = 2$. By the triangle free property there does not exist edge between two different neighbours of x. There is at least one vertex $z \in I(x,y)$, so there must exist its diametrical pair, say w. Now, if there were any other vertex in $I(x,y)$, then z would have more than one diametrical pair, a contradiction. ∎

Step 2. $I(u,\alpha) \cap I(\bar{\alpha},v) = \emptyset$.

Proof. Suppose in contrary that there exists β of level i in $I(u,v)$ such that $\beta \in I(u,\alpha) \cap I(\bar{\alpha},v)$. Then we have $d(\bar{\alpha},\beta) = i-1$ and $d(\beta,\alpha) = d-i$ so

$$d(\alpha,\bar{\alpha}) \leq d(\bar{\alpha},\beta) + d(\beta,\alpha) = i-1+d-i < d$$

would hold, a contradiction. ∎

Step 3. *There does not exist an $x \in (I(u,v) \setminus (I(u,\alpha) \cup I(\bar{\alpha},v)))$ of level i such that it has a neighbour of level $i-1$ in $I(u,\alpha)$, and one of level $i+1$ in $I(\bar{\alpha},v)$.*

Proof. Suppose that there exists such an x and let its neighbours be z and w ($l(z) = i-1$ and $l(w) = i+1$). Then $d(z,w) = 2$, so $I(z,w) \cong \mathcal{B}_2$. Hence, there exists a vertex t such that $t \in I(z,w) \setminus \{x\}$. It is easy to see that t is of level i in $I(u,v)$. Suppose first, that $w \neq v$ and $z \neq u$. If $t \notin I(\bar{\alpha},v)$, then w would have at least $i+2$ neighbours of level i in $I(u,v)$ that contradicts to the induction hypothesis $I(u,w) \bowtie \mathcal{B}_{i+1}$. On the other hand, if $t \in I(\bar{\alpha},v)$, then z has at least $d-i+2$ neighbours on level i. If $z \neq u$, then this contradicts to the induction hypothesis. Now let $w = v$. Pick a neighbour q of v in $I(\bar{\alpha}, v)$. Let a be a common neighbour of x and q on level $d-2$, and let b be that of α and q provided by the quadrangle property. Then $q \notin I(u,\alpha) \cup I(\bar{\alpha}, v)$ and $b \in I(u,\alpha)$. So q has at least d neighbours on level $d-2$ in $I(u,v)$ that contradicts to the induction hypothesis $I(u,q) \bowtie \mathcal{B}_{d-1}$. For $z = u$ we obtain contradiction in similar way as in the case $w = v$. ∎

Step 4. *If $l(y) = j \neq d$ and all neighbours of y of level $j-1$ are in $I(u,v) \setminus I(u,\alpha)$, then $I(u,\alpha) \cap I(u,y) = \{u\}$.*

Proof. Suppose that $z \neq u$, such that $z \in I(u,\alpha) \cap I(u,y)$ and z is of the highest level subject to this condition. Let $l(z) = k$. Then z has $d-1-k$ neighbours of level $k+1$ in $I(u,\alpha)$. Furthermore, z has $j-k$ neighbours of level $k+1$ in $I(u,y)$. All these are different, so z has at least $d+j-2k-1 > d-k$ neighbours of level $k+1$ in $I(u,v)$ that contradicts to the induction hypothesis $I(z,v) \bowtie \mathcal{B}_{d-k}$. ∎

Note 1. *Similar proposition holds if we consider level $j+1$ neighbours with the conclusion that $I(\bar{\alpha}, v) \cap I(y,v) = \{v\}$.*

Step 5. *If $z \in I(u,\alpha)$, $z \neq u$, $l(z) = i$, $w \in I(u,v) \setminus I(u,\alpha)$, $l(w) = i+1$ and $\{z,w\}$ is an edge, then $w \in I(\bar{\alpha}, v)$. The same holds for changing $I(u,\alpha)$ to $I(\bar{\alpha}, v)$ and $i+1$ to $i-1$.*

Proof. Suppose that $w \notin I(\bar{\alpha}, v)$. Then no neighbour of w of level $i+2$ can be in $I(\bar{\alpha}, v)$ by Step 3. Thus, by Step 4 we have that $I(\bar{\alpha}, v) \cap I(w,v) = \{v\}$. Let $w = t_0, t_1, \ldots, t_s = v$ be a shortest ($d-i-1$-long) path from w to v. Because $I(u, t_{s-1}) \cap I(u,\alpha) \ni z \neq u$, t_{s-1} must have a neighbour of level $d-2$ in $I(u,\alpha)$. This contradicts to Step 3. ∎

Note 2. *If $z \in I(u,\alpha)$ ($z \in I(\bar{\alpha}, v)$) such that $l(z) = i$, then all neighbours of z of level $i-1$ ($i+1$) are in $I(u,\alpha)$ ($I(\bar{\alpha}, v)$).*

Step 6. *For every $z \in I(u,\alpha)$, such that $z \neq u$ and $l(z) = i$, there exists exactly one w $l(w) = i+1$ such that $\{z,w\}$ is an edge and $w \notin I(u,\alpha)$. The same holds for exchanging $I(u,\alpha)$ to $I(\bar{\alpha}, v)$ and $i+1$ to $i-1$.*

Proof. $I(z,v) \bowtie \mathcal{B}_{d-i}$ by the induction hypothesis ($z \neq u$), so z has $d-i$ neighbours of level $i+1$ in $I(u,v)$. Out of those $d-i-1$ lie in $I(u,\alpha)$. ∎

Let us denote the above w by z' and let $u' = \bar{\alpha}$ by definition. Then by Step 3-5, $z' \in I(\bar{\alpha}, v)$ if $z \in I(u,\alpha)$ and vica versa. Furthermore, $(z')' = z$.

Hence, $z \mapsto z'$ is a bijection between $I(u,\alpha)$ and $I(\bar{\alpha},v)$.

Step 7. *$z \mapsto z'$ is an isomorphism between the graphs induced by edges between different levels of $I(u,\alpha)$ and $I(\bar{\alpha},v)$, respectively.*

Proof. It is enough to show that $\{x,y\}$ is an edge in $I(u,\alpha)$ implies $\{x',y'\}$ is an edge in $I(\bar{\alpha},v)$. The reverse comes from $(z')' = z$. Let $\{x,y\}$ be an edge, such that $l(x) = i$ and $l(y) = i+1$. Then $d(x,y') = d(x',y) = 2$, so x and y' must have a unique common neighbour other than y. If it is in $I(\bar{\alpha},v)$, then by Step 6, it must be x'. If it is in $I(u,\alpha)$, then it must be on level $i+1$. Then again by Step 6, it must be y, a contradiction. If it is in $I(u,v) \setminus (I(u,\alpha) \cup I(\bar{\alpha},v))$, then it contradicts to Step 3. ■

Step 8. *If $l(z) = i$, $z \in I(u,\alpha)$, $l(w) \leq i$ and $w \in I(\bar{\alpha},v)$, then $\{z,w\}$ is not an edge.*

Proof. By the triangle inequality we have that

$$\begin{aligned}(\alpha,\bar{\alpha}) &\leq d(\alpha,z) + d(z,w) + d(w,\bar{\alpha}) \\ &\leq d-1-i+1+i-1 = d-1\end{aligned}$$

a contradiction. ■

Now, if we do not consider edges between vertices of the same level, then $I(u,\alpha)$ and $I(\bar{\alpha},v)$ are $d-1$ dimensional hypercubes, respectively. Furthermore, the latter one is positioned one level higher, and there is an edge between corresponding vertices of $I(u,\alpha)$ and $I(\bar{\alpha},v)$. No other edge is going between vertices of different levels so we can conclude that $I(u,\alpha) \cup I(\bar{\alpha},v) \bowtie \mathcal{B}_d$.

Step 9. $I(u,v) = I(u,\alpha) \cup I(\bar{\alpha},v)$.

Proof. Suppose in contrary that there exists a neighbour x of v that is in $I(u,v) \setminus (I(u,\alpha) \cup I(\bar{\alpha},v))$. Then by the quadrangle property there exists $z \in I(u,v)$ of level $d-2$ such that z is connected to both x and α. This implies that $z \in I(u,\alpha)$. However, this contradicts to Step 3. ■

Proof of Theorem 1.8. Let $u,v \in V(G)$ and $x,y \in I(u,v)$. Let us denote the distance of x and y in the graph $I(u,v)$ by $d_{uv}(x,y)$ to distinguish from their distance in G, $d(x,y)$. If $d_{uv}(x,y) = d(x,y) = b$, then $I(x,y) \bowtie \mathcal{B}_b$. However, $I(u,v)$ contains a b-cube between x and y because $I(u,v) \bowtie \mathcal{B}_d$.

This means that $I(x,y) \subset I(u,v)$. On the other hand, if $d_{uv}(x,y) > d(x,y)$, then x does not have a diametrical pair in $I(u,v)$, which contradicts to the assumption that G is spherical. ∎

References

[1] I. Havel and P. Liebl, unpublished result (Prague).

[2] H. M. Mulder, *The Interval Function of a Graph*, Math. Centre Tracts, **132** Amsterdam 1980.

Sergey L. Bezrukov
Inst. for Problems of Information Transmission, Ermolovoi str. 19. 101447, Moscow, USSR

Attila Sali
Mathematical Institute of the Hungarian Academy of Sciences H-1364 Hungary Budapest, P.O.B. 127