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On the Maximum Densities of j-th Smallest Distances

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ABSTRACT

Let $s_j(n)$ denote the maximum number of occurrences of the j-th smallest distance in a set of n points in the plane, and let $c_j := \sup_n \frac{s_j(n)}{n}$ denote its maximum density. It is already known that $c_1 = 3$ and $c_2 = 24/7$. We prove asymptotic upper bounds for the c_j, show that $c_j = O(j^{\frac{3}{4}})$ and give bounds for c_3 and c_4.

1. Introduction

In 1946 Erdős posed the problem of determining the maximum number of unit distances in sets of n points in the plane [5]. This still unsolved problem ([1],[4],[7],[8]) led to the study of maximum multiplicities of special distances in special configurations. Harborth proved in 1974 that the smallest distance in a set of n points in the plane occurs at most $\lfloor 3n - \sqrt{12n-3} \rfloor$ times [6]. Let $s_j(n)$ denote the maximum number of occurrences of the j-th smallest distance among n points in the plane, and $c_j := \sup_n \frac{s_j(n)}{n}$. Vesztergombi showed $c_j \leq 3j$ and $\frac{24}{7} \leq c_2 \leq 5$ [9], which I improved to $c_2 = \frac{24}{7}$ [2]. The best known general lower bound for c_j is of order $j^{\frac{c}{\log \log j}}$ (the corresponding pointsets are subsets of the triangular lattice), and it seems improbable that any significant improvement can be made on this. The aim of this paper is therefore to improve the upper bounds on c_j, which is done using a geometrical lemma and linear optimization theory.

2. The results

Let $h(n)$ denote the maximum number of unit distances among n points in the plane. Then we have the following theorem:

Theorem 1. *If $h(n) = O(n^{1+\varepsilon})$ holds for some $\varepsilon \in]0,1[$, then $c_j = O\left(j^{\frac{1}{2(1-\varepsilon)}}\right)$ holds.*

Corollary.

$$c_j = O\left(j^{\frac{3}{4}}\right)$$

follows, since $h(n) = O(n^{\frac{4}{3}})$ was proved in [4] and [8].

Theorem 2.

$$\frac{32}{9} \le c_3 \le \frac{27}{4} \quad , \qquad 6 \le c_4 \le \frac{37}{4}$$

The proof of the theorems is based on the following lemma:

Lemma 1. *Let $f(c,n)$ denote the minimum number of distances smaller than c among n points on a unit circle. If $\frac{2\pi}{k+1} \le 2\arcsin\frac{c}{2} < \frac{2\pi}{k}$, then*

$$f(c,n) = \begin{cases} \sum_{i=0}^{k-1} \binom{\lfloor \frac{n+i}{k} \rfloor}{2} & \text{if } c \ne 2 \text{ and } n \ne k+1 \\ & \text{or if } c \ne 2,\ n = k+1 \text{ and } 2\arcsin\frac{c}{2} \ne \frac{2\pi}{k+1} \\ 0 & \text{if } c \ne 2,\ n = k+1 \text{ and } 2\arcsin\frac{c}{2} = \frac{2\pi}{k+1} \\ \binom{n}{2} - \lfloor \frac{n}{2} \rfloor & \text{if } c = 2 \end{cases}$$

Constructions for the extremal pointsets are given in the proof of the lemma.

We use only the special case $c = 1$ of this lemma, which states:
Among n points on a unit circle at least

$$f(n) := f(1,n) = \begin{cases} 0 & \text{if } n \le 6 \\ \lceil \frac{1}{10}n^2 - \frac{1}{2}n - \frac{1}{5} \rceil & \text{otherwise} \end{cases}$$

distances smaller than 1.

3. Proof of the theorems

Fix j and n and consider a set S of n points in the plane with maximum number of j-th smallest distances. Let $d_1 < d_2 < \cdots$ denote the distances among points of S (without loss of generality we assume $d_j = 1$) with d_i occurring s_i times. Denote further with a_i the number of points in S with exactly i points of S at distance d_j (at most $6j$ points are possible [9]). By use of the special case of Lemma 1, we get, since each pair of points lies on at most 2 circles with radius d_j, the following inequality

$$2\,(s_1 + \cdots + s_{j-1}) \geq \sum_{i=0}^{6j} f(i) a_i\,. \tag{1}$$

Denote with $g(j)$ the maximum number of points in any set S with distance at most d_j to a fixed point of S. By counting endpoints of distances at most d_{j-1} we get

$$g(j-1)n \geq 2\,(s_1 + \cdots + s_{j-1})\,. \tag{2}$$

Subtracting (2) from (1) and using $n = \sum_{i=0}^{6j} a_i$, we get

$$\sum_{i=0}^{6j} (f(i) - g(j-1)) a_i \leq 0, \tag{3}$$

a linear inequality for the $(a_i)_{i=0}^{6j}$, which describes, together with $a_i \geq 0$ for all i and $\sum_{i=0}^{6j} a_i = n$, a polyhedron for the $(a_i)_{i=0}^{6j}$. Since $\sum_{i=0}^{6j} i a_i = 2 s_j(n)$ holds for the $(a_i)_{i=0}^{6j}$ corresponding to our set S with maximum number of j-th smallest distances, we have the upper bound

$$s_j(n) \leq \frac{1}{2} \max \left\{ \sum_{i=0}^{6j} i a_i \;\middle|\; \sum_{i=0}^{6j} a_i = n,\; \sum_{i=0}^{6j} (f(i) - g(j-1)) a_i \leq 0,\; a_i \geq 0 \text{ for all } i \right\}.$$

Since all restrictions are linear in n, we get

$$c_j \leq \frac{1}{2} \max \left\{ \sum_{i=0}^{6j} i a_i \;\middle|\; \sum_{i=0}^{6j} a_i = 1,\; \sum_{i=0}^{6j} (f(i) - g(j-1)) a_i \leq 0,\; a_i \geq 0 \text{ for all } i \right\}. \tag{4}$$

Proof of Theorem 2. For fixed j we can solve the linear optimization Problem (4) exactly (e.g. by the simplex method [3]). For $j = 3$ we use that Vesztergombi [9] proved $g(2) = 12$, so we have

$$c_3 \leq \frac{27}{4} = \frac{1}{2} \max \left\{ \sum_{i=0}^{18} i a_i \;\middle|\; \sum_{i=0}^{18} (f(i) - 12) a_i \leq 0, \right. \\ \left. \sum_{i=0}^{18} a_i = 1, a_i \geq 0 \text{ for all } i \right\}.$$

By this follows $2(s_1 + s_2 + s_3) \leq (g(2) + 2c_3)n \leq \frac{51}{2} n$, so we have also proved

$$c_4 \leq \frac{37}{4} = \frac{1}{2} \max \left\{ \sum_{i=0}^{24} i a_i \;\middle|\; \sum_{i=0}^{24} (f(i) - \tfrac{51}{2}) a_i \leq 0, \right. \\ \left. \sum_{i=0}^{24} a_i = 1, a_i \geq 0 \text{ for all } i \right\}.$$

The lower bounds of Theorem 2 follow from subsets of the point-sets in Figure 2. ■

Proof of Theorem 1. Take a set S consisting of a point P and $g(j)$ points with distance at most d_j to P. Take a disk with radius greater than d_j and center P, this disk contains S. Then there is a $\frac{\pi}{3}$-sector of this disk which contains at least $1 + \frac{1}{6} g(j)$ points (including P), among which only distances $d_1, \ldots, d_j$ occur. By the pigeonhole principle, we have one distance occurring at least $\frac{1}{2j} \left(1 + \frac{1}{6} g(j)\right) \left(\frac{1}{6} g(j)\right)$ times, which gives a lower bound for h:

$$h \left(1 + \frac{1}{6} g(j)\right) \geq \frac{1}{2j} \left(1 + \frac{1}{6} g(j)\right) \left(\frac{1}{6} g(j)\right).$$

The theorem assumed an upper bound of $h(n+1) \leq cn^{1+\varepsilon}$, together this implies

$$g(j) < 6 \, (2cj)^{\frac{1}{1-\varepsilon}}. \tag{5}$$

To solve the family of linear optimization problems which arise from (4) and (5) we use that

$$(f(k) - g(j-1)) \geq \frac{1}{10} k^2 - \frac{1}{2} k - \frac{1}{5} - 6(2c(j-1))^{\frac{1}{1-\varepsilon}}$$

where the right side is convex in k, and apply the following lemma:

Lemma 2. *Let $w : \{0, 1, \ldots, n\} \to \mathbb{R}$ be convex, $w(0) < 0$, $w(n) > 0$, and* $\mathrm{intpol}_{\mathrm{bl}}$, $\mathrm{intpol}_{\mathrm{cf}}$ *be the broken line and an arbitrary convex function, both of which interpolate w on $\{0, 1, \ldots, n\}$. Then*

$$\max\left\{\sum_{k=0}^{n} ka_k \;\middle|\; \sum_{k=0}^{n} w(k)a_k \leq 0, \sum_{k=0}^{n} a_k = 1, a_k \geq 0 \text{ for all } k\right\}$$
$$= \mathrm{intpol}_{\mathrm{bl}}^{-1}(0) \leq \mathrm{intpol}_{\mathrm{cf}}^{-1}(0)$$

holds.

By this lemma, the positive zero of $\frac{1}{10}x^2 - \frac{1}{2}x - \frac{1}{5} - 6(2cj)^{\frac{1}{1-\varepsilon}}$ is an upper bound for $2c_j$, which is $O(j^{\frac{1}{2(1-\varepsilon)}})$. ■

4. Proof of the lemmas

Proof of Lemma 1. The case $c = 2$ is trivial, since two points on the unit circle have a distance smaller than 2 iff they are not diametrically opposite; so we always assume $c < 2$.

In the following we will use angular distances between the points on the unit circle: two points have a distance smaller than c iff their angular distance is smaller than $\alpha := 2\arcsin\frac{c}{2}$ ($\frac{2\pi}{k+1} \leq \alpha < \frac{2\pi}{k}$ by assumption).

To show that the given function is an upper bound for f, consider n points arranged in k groups of $\lfloor\frac{n+i}{k}\rfloor$ points ($i = 0, \ldots, k-1$) around the vertices of an inscribed regular k-gon of the unit circle, such that the distances between different groups are all at least c; this can be improved in the case $n = k+1$, $\alpha = \frac{2\pi}{k+1}$ by taking the vertices of an inscribed regular $k+1$-gon.

Next we prove that the function gives the right values for $k+2 \leq n \leq 2k+1$. Denote the angular distances between consecutive points on the unit circle with $\phi_1, \ldots, \phi_n$. Since $\sum_{i=1}^{n} \phi_i = 2\pi$ and all ϕ_i are positive, at most k of the ϕ_i are greater than α, the remaining $n-k$ therefore correspond to $n-k$ distances smaller than c. In the case of $n = 2k+1$, two of the remaining ϕ_i must be consecutive, and since their sum is smaller than $2\pi - k\alpha \leq \frac{2\pi}{k+1} \leq \alpha$, their sum corresponds to a further distance smaller than c. So we have $f(c, n) \geq n-k$ for $k+2 \leq n \leq 2k$ and $f(c, 2k+1) \geq k+2$, which equals the given upper bound.

Finally we prove the recursive inequality $f(c, n+k) \geq f(c, n) + n$ for $n \geq k+2$. Assume we are given $n+k$ points on the unit circle with minimum number of distances smaller than c. Select k of these points by the following procedure: Fix P_1 arbitrarily. Denote with P_i $(2 \leq i \leq k)$ the first point of the given set which follows P_{i-1} in positive orientation and has an angular distance of at least α to P_{i-1}. Either each of the remaining n points has a distance smaller than c to one of the k points (so removing them gives a set of n points with at most $f(c, n+k) - n$ distances smaller than c), or $\alpha = \frac{2\pi}{k+1}$ and there is a point P_{k+1} such that $P_1, \ldots, P_{k+1}$ form a regular $k+1$-gon. In this case each of the remaining $n-1$ points has a distance smaller than c to two of the P_i, so removing only k of them we get a set of n points with at most $f(c, n+k) - \left\lceil \left(2 - \frac{2}{k+1}\right)(n-1) \right\rceil \leq f(c, n+k) - n$ (for $n \geq k+2$, $k \geq 2$, since $k = 1$ in this case implies $c = 2$) distances smaller than c. Since this recursion holds with equality for the upper bound, Lemma 1 is proved. ■

Proof of Lemma 2. Since the polyhedron

$$\mathbb{P} := \left\{ (a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{\nu=0}^{n} w(\nu) a_\nu \leq 0, \sum_{\nu=0}^{n} a_\nu = 1, a_\nu \geq 0 \text{ for all } \nu \right\}$$

is n-dimensional with $n+2$ restricting hyperplanes ($n+1$ nonnegativity constraints and $\sum_{\nu=0}^{n} w(\nu) a_\nu \leq 0$), in any vertex all but at most two of the a_ν are zero. Without the constraint $\sum_{\nu=0}^{n} w(\nu) a_\nu \leq 0$, the optimal solution would be $a_n = 1$, $a_\nu = 0$ else, which is not a point of $\mathbb{P}$ ($w(n) > 0$), so we can assume $\sum_{\nu=0}^{n} w(\nu) a_\nu = 0$. So the optimal vertex contains exactly two non-zero variables a_μ and a_ν $(\mu < \nu)$.

If $\nu \neq \mu + 1$, then there is some integer index $\kappa = \lambda\mu + (1-\lambda)\nu$ with $\mu < \kappa < \nu$ and we can define a new point $\hat{a} \in \mathbb{R}^{n+1}$ by $\xi := \min\left(\frac{a_\mu}{\lambda}, \frac{a_\nu}{1-\lambda}\right)$, $\hat{a}_\mu := a_\mu - \lambda\xi$, $\hat{a}_\kappa := \xi$, $\hat{a}_\nu := a_\nu - (1-\lambda)\xi$ and $\hat{a}_\iota := 0$ else. We have $\hat{a} \in \mathbb{P}$, since $\hat{a}_\mu, \hat{a}_\nu, \hat{a}_\kappa \geq 0$, $\hat{a}_\mu + \hat{a}_\nu + \hat{a}_\kappa = a_\mu + a_\nu = 1$ and

$$\begin{aligned} & w(\mu)\hat{a}_\mu + w(\kappa)\hat{a}_\kappa + w(\nu)\hat{a}_\nu \\ &= w(\mu)a_\mu + w(\nu)a_\nu - \xi\Big(\lambda w(\mu) + (1-\lambda)w(\nu) - w(\lambda\mu + (1-\lambda)\nu)\Big) \\ &\leq w(\mu)a_\mu + w(\nu)a_\nu \leq 0\,, \end{aligned}$$

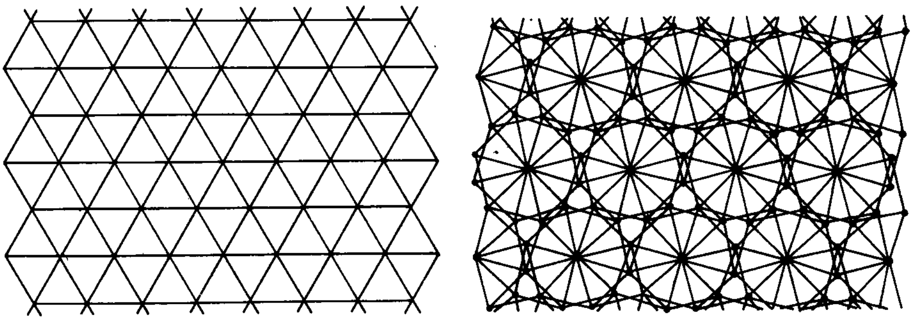

Fig. 1.. *The extremal point-sets for the smallest and second-smallest distance*

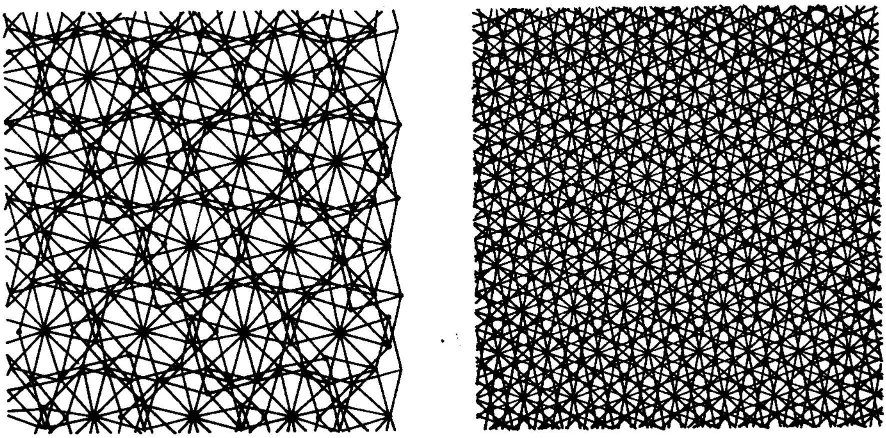

Fig. 2.. *The best known point-sets for the third- and fourth-smallest distance*

furthermore $\hat{a}$ is vertex of $\mathbb{P}$, since one of $\hat{a}_\mu, \hat{a}_\nu$ is zero, and finally $\hat{a}$ is optimal, since $\mu\hat{a}_\mu + \nu\hat{a}_\nu + \kappa\hat{a}_\kappa = \mu a_\mu + \nu a_\nu$. So there is an optimal vertex with nonzero variables $a_\kappa, a_{\kappa+1}$. In this case $w(\kappa)a_\kappa + w(\kappa+1)a_{\kappa+1} = 0$, $a_\kappa + a_{\kappa+1} = 1$ holds, so the optimal value $\kappa a_\kappa + (\kappa+1)a_{\kappa+1} = \kappa + \frac{w(\kappa)}{w(\kappa)-w(\kappa+1)}$ equals the zero of the linear function which interpolates w on $\{\kappa, \kappa+1\}$, which is the only zero of the broken line which interpolates w on $\{0, 1, \ldots, n\}$.

Since the interpolating broken line is the maximum of all interpolating convex functions, and w is increasing on $\kappa, \kappa+1$, the zero of the interpolating broken line is a lower bound for the zero of any interpolating convex function. This proves Lemma 2. ■

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