COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
60. SETS, GRAPHS AND NUMBERS, BUDAPEST (HUNGARY), 1991

Extreme Hypermetrics and L-polytopes

M. DEZA, V. P. GRISHUKHIN and M. LAURENT

ABSTRACT

Hypermetric spaces are in correspondence with L-polytopes in lattices ([2],[10]). Using this connection, we define the rank of an L-polytope as the rank of the corresponding hypermetric space, i.e. the dimension of the smallest face of the hypermetric cone that contains it. We study some properties for this notion of rank, in particular, its invariance, some bounds for it, also an additive formula for the rank of reducible L-polytopes. We consider especially the case of L-polytopes of rank 1: extreme L-polytopes that correspond to extreme rays of the hypermetric cone. In particular, the two Gosset polytopes in the root lattices E_6, E_7 are given as examples of extreme L-polytopes, yielding a class of extreme rays of the hypermetric cones on 7, 8 points. We also construct several examples of extreme L-polytopes of dimension 15, 16 coming from the Barnes-Wall lattice Λ_{16} and of dimension 22, 23 coming from the Leech lattice Λ_{24}.

1. Introduction

The hypermetric cone H_{n+1} is the set of all vectors $d = (d_{ij})_{0\leq i<j\leq n} \in \mathbb{R}^{\binom{n+1}{2}}$ that satisfy the following inequalities (1.1), called hypermetric inequalities.

$$\sum_{0\leq i<j\leq n} z_i z_j d_{ij} \leq 0 \quad \text{for } z \in \mathbb{Z}^{n+1},\ \sum_{i=0}^{n} z_i = 1 \tag{1.1}$$

For $z \in \mathbb{Z}^{n+1}$ whose components are: $z_i = z_j = 1, z_k = -1, z_h = 0$ for $0 \le h \le n, h \ne i, j, k$, the inequality (1.1) is, in fact, the triangle inequality: $d_{ij} - d_{ik} - d_{jk} \le 0$. Hence, the hypermetric cone H_{n+1} is a subcone of the metric cone M_{n+1},

$$M_{n+1} = \left\{ d \in \mathbb{R}^{\binom{n+1}{2}} \ : \ d_{ij} - d_{ik} - d_{jk} \le 0 \text{ for } i, j, k \in \{0, 1, .., n\} \right\}$$

On the other hand, a proper subcone of H_{n+1} is the cut cone C_{n+1} defined as the cone generated by all cut metrics d_S for $S \subseteq \{0, 1, \dots, n\}$, where $(d_S)_{ij} = 1$ if $|\{i, j\} \cap S| = 1$ and $(d_S)_{ij} = 0$ otherwise, for $0 \le i < j \le n$. Note that the cut metrics d_S and $d_{\{0,1,\dots,n\}-S}$ coincide.

The cones M_{n+1} and C_{n+1} are clearly polyhedral. It was proved in [10] that H_{n+1} too is a polyhedral cone. In this paper, we develop some tools for studying the extreme rays of H_{n+1}. It is well known that all the $2^n - 1$ nonzero cut metrics d_S which generate C_{n+1} define extreme rays of M_{n+1} and thus of H_{n+1}.

Let g denote the linear bijective transformation of $\mathbb{R}^{\binom{n+1}{2}}$ defined by $g(d) = a$ for $d = (d_{ij})_{0 \le i < j \le n}$, $a = (a_{ij})_{1 \le i \le j \le n}$ satisfying:

$$\begin{cases} d_{0i} = a_{ii} & \text{for } 1 \le i \le n \\ d_{ij} = a_{ii} + a_{jj} - 2a_{ij} & \text{for } 1 \le i < j \le n \end{cases}$$

The image of the hypermetric cone H_{n+1} under g is the cone:

$$g(H_{n+1}) = \left\{ a \in \mathbb{R}^{\binom{n+1}{2}} : \sum_{1 \le i,j \le n} z_i z_j a_{ij} - \sum_{i=1}^{n} z_i a_{ii} \ge 0 \text{ for } z \in \mathbb{Z}^n \right\}.$$

In his very interesting paper, Erdahl ([12]) considers the cone:

$$P_n = \left\{ (a, b, c) \in \mathbb{R}^{\binom{n}{2}+n+1} \ : \sum_{1 \le i,j \le n} z_i z_j a_{ij} + \sum_{i=1}^{n} b_i z_i + c \ge 0 \text{ for } z \in \mathbb{Z}^n \right\}.$$

Hence, $g(H_{n+1}) = P_n \cap \{(a, b, c) : \ c = 0 \text{ and } b_i = -a_{ii} \text{ for } 1 \le i \le n\}$. Therefore, the hypermetric cone is (via the map g) a polyhedral section of the cone P_n. Erdahl shows that P_n has three types of extreme rays; actually, one of them (which he calls perfect) corresponds to the extreme rays of the hypermetric cone.

We exploit in this paper, as well as in the previous paper ([10]), the following correspondence between hypermetric spaces and L-polytopes.

Let L be a lattice in $\mathbb{R}^k$ and let S be an empty sphere (or hole) in L, i.e. the ball with boundary sphere S contains no lattice point in its interior, but the lattice points lying on S have full rank k. Then, the convex hull of the lattice points lying on S is called an L-polytope in L (or Delaunay polytope, or constellation).

It was proved ([2], [10]) that, for any hypermetric $d \in H_{n+1}$, there exist a lattice L_d in $\mathbb{R}^k, k \leq n$, an L-polytope P_d in L_d and a generating map ϕ_d from $X = \{0, 1, \ldots, n\}$ to the set of vertices of P_d (i.e. $\phi_d(X)$ generates L_d) such that $d_{ij} = \|\phi_d(i) - \phi_d(j)\|^2$ for $0 \leq i < j \leq n$, these objects being defined uniquely up to orthogonal transformation. Hence, the points of H_{n+1} correspond to L-polytopes; if two points lie on the interior of a common face of H_{n+1}, then their associated L-polytopes are affinely equivalent.

Conversely, if P is an L-polytope with set of vertices $V(P)$ and if we set $d_0(u, v) = \|u - v\|^2$ for $u, v \in V(P)$, then d_0 is a hypermetric on $V(P)$.

Given a hypermetric $d \in H_{n+1}$, its rank $r(d)$ is the dimension of the smallest face of H_{n+1} that contains d. Hence, hypermetrics of rank 1 define extreme rays of H_{n+1}. If P is an L-polytope, the rank of the hypermetric space $(V(P), d_0)$ is also called the rank of P and denoted by $r(P)$. So, extreme L-polytopes, i.e. L-polytopes of rank 1, correspond to extreme rays of the hypermetric cone.

This notion of rank and, in particular, extreme L-polytopes are central concepts that we study in this paper. Let us now describe the contents of the paper.

In section 2.1, we recall all necessary definitions for L-polytopes and lattices and, in section 2.2, the connection between hypermetric spaces and L-polytopes. We give in section 2.3 some first basic results.

In section 3, we introduce the rank of a hypermetric space and observe that it is, in fact, an invariant of the associated L-polytope.

In section 4, we define the rank $r(P)$ of an L-polytope P and prove that $r(P)$ is, in fact, equal to the rank of the hypermetric space (V, d_0) for any generating subset $V \subseteq V(P)$ of L_d. We give an explicit description of the system $\mathcal{S}(V, d_0)$ consisting of all the hypermetric inequalities on V that are satisfied at equality by the hypermetric space (V, d_0); it is useful for the practical computation of the rank of P, since $r(P)$ is equal to the dimension of the solution set to the system $\mathcal{S}(V, d_0)$.

We show in section 5 an additivity property for the rank of an L-polytope, namely, $r(P) = r(P_1) + r(P_2)$ if $P = P_1 \times P_2$ is the L-polytope obtained by direct product of two L-polytopes P_1, P_2.

In section 6, we consider basic L-polytopes, i.e. L-polytopes for which one can find a subset of their vertices that is an affine basis of the lattice; we give some upper and lower bounds for the rank of a basic L-polytope in terms of its number of vertices and its rank.

In section 7, we consider extreme L-polytopes, i.e. L-polytopes of rank 1, which correspond to extreme rays of the hypermetric cone. The bounds from section 6 yield lower bounds for the number of vertices of an extreme basic L-polytope. There is a striking analogy between these bounds and some known upper bounds for the number of points in a spherical two-distance set and for the number of lines in an equiangular set of lines. Through this observation, we have several candidates for extremality: the Gosset polytopes 2_{21} and 3_{21} of dimension 6 and 7, respectively, coming from the root lattice E_8 and some polytopes of dimension 22 and 23 coming from the Leech lattice Λ_{24}.

We show in section 8 that the two Gosset polytopes are indeed extreme and we give, in section 9, some more detailed information on the extreme rays of H_7 that are derived from the Gosset polytope 2_{21}. Actually, the two Gosset polytopes 2_{21} and 3_{21} are the only extreme L-polytopes arising from root lattices (besides the 1-dimensional segment $[0, 1]$ that corresponds to the extreme rays defined by the cut metrics).

In section 10, we give two examples of extreme L-polytopes of dimension 22 and 23 coming from the Leech lattice Λ_{24}; they are related, respectively, to some spherical two-distance set and to some equiangular set of lines and they are both tight for the lower bounds on the number of vertices.

In section 11, we give several examples of extreme L-polytopes of dimension 15, 16, coming from the Barnes-Wall lattice Λ_{16}; some of them are also tight for our lower bounds, but they are not related to some spherical 2-distance set or some equiangular set of lines.

2. Preliminaries

2.1. Lattices and L-polytopes

Given $x, y \in \mathbb{R}^k$, $x \cdot y = \sum_{i=1}^{k} x_i y_i$ denotes the scalar product of x, y, $\|x\| = \sqrt{x \cdot x}$ denotes the Euclidean norm of x and we set:

$$d_0(x, y) = \|x - y\|^2 = \sum_{i=1}^{k} (x_i - y_i)^2 \tag{2.1}$$

A subset L of $\mathbb{R}^k$ is called a *lattice* if L is a discrete subgroup of $\mathbb{R}^k$. A subset $\{v_1, \ldots, v_m\}$ of L is said to be *generating* for L if, for every $v \in L$, there exist some integers $z_1, \ldots, z_m$ such that:

$$v = \sum_{i=1}^{m} z_i v_i \tag{2.2}$$

If, furthermore, there is unicity of the integers $z_1, \ldots, z_m$ in (2.2), i.e. the set $B = \{v_1, \ldots, v_m\}$ is linearly independent, then B is called a (linear) *basis* of L and $m = |B|$ is the *dimension* of the lattice L.

A subset L' of $\mathbb{R}^k$ is called an *affine lattice* if L' is the image of a lattice by some translation, i.e. $L' = L + a = \{v + a : \ v \in L\}$ for some lattice L and some $a \in \mathbb{R}^k$. In other words, if L is a lattice in $\mathbb{R}^k$ (so, $0 \in L$) and if one changes the origin, say $-a \in \mathbb{R}^k$ is the new origin, then L becomes the affine lattice $L + a$ in the new system of coordinates with origin $-a$.

A subset $A = \{v_0, v_1, \ldots, v_m\}$ of an affine lattice L' is called an *affine generating set* for L' (resp. an *affine basis* of L') if, for every $v \in L'$, there exist some integers (resp. a unique system of integers) $z_0, z_1, \ldots, z_m$ such that

$$v = \sum_{i=0}^{m} z_i v_i \quad \text{and} \quad \sum_{i=0}^{m} z_i = 1 \tag{2.3}$$

If A is an affine basis of L', then $m = |A| - 1$ is the dimension of L'. For example, if $\{v_1, \ldots, v_m\}$ is a (linear) basis of the lattice L, then the set $\{a, v_1 + a, \ldots, v_m + a\}$ is an affine basis of the affine lattice $L + a$.

In what follows, we shall use, for simplicity, the same word "lattice" for denoting both a usual lattice (i.e. containing 0) and an affine lattice (i.e.

translate of a lattice). Also, it will be more convenient for our treatment to work with affine bases of lattices rather than linear bases. When we say that a set is generating for a lattice L, we mean that it is an affine generating set for L.

If $B = \{v_0, v_1, \ldots, v_m\}$ is a set of vectors of $\mathbb{R}^k$, the matrix $\text{Gram}(B) = BB^T$ is called the *Gram matrix* of B; it is a $(m+1) \times (m+1)$ matrix whose (i,j)-th element is $v_i \cdot v_j$. Let $B = \{v_0, v_1, \ldots, v_m\}$ be an affine basis of the lattice L in $\mathbb{R}^k$ and set $B' = \{v_1 - v_0, v_2 - v_0, \ldots, v_m - v_0\}$. Then, the quantity $\det(\text{Gram}(B'))$ is an invariant of L, called *determinant* of L and denoted by $\det(L)$. In particular, if L is a lattice of dimension k in $\mathbb{R}^k$ (i.e. $n = k$), then $\det L = |\det(B')|^2$ holds.

Let L be a k-dimensional lattice in $\mathbb{R}^k$ and let S be a sphere in $\mathbb{R}^k$ of center c and radius r, $S = \{x \in \mathbb{R}^k : \|x - c\| = r\}$. Then, S is called an *empty sphere* (or *hole*) in L if $\|v - c\| \geq r$ holds for all $v \in L$ and the set $S \cap L$ has rank k (i.e. affine rank $k+1$), i.e. no lattice points are lying in the interior of the ball with boundary sphere S but S is completely determined by the lattice points lying on it. The empty sphere S is called *generating* if $S \cap L$ generates the lattice L. If S is an empty sphere in L, then the set $S \cap L$ is finite; the convex hull of $S \cap L : P = \text{conv}(S \cap L)$ is called an *L-polytope* (or *Delaunay polytope,* or *constellation*) in the lattice L. The center of the sphere S is also called the center of the polytope P.

If P is an L-polytope in a lattice L, a subset V of its set of vertices $V(P)$ is called *generating* if V generates L and V is called *basic* if V is an affine basis of L. One says that P is a *generating L-polytope* in L if its set of vertices $V(P)$ generates L, i.e. if one can find a subset of $V(P)$ which is generating. The L-polytope P is called *basic* if there exists an affine basis B of L such that $B \subseteq V(P)$. For instance, the root lattice E_8 contains no generating L-polytope. Actually, we are interested here only in generating L-polytopes. Indeed, as we recall in section 2.2, hypermetric spaces are associated with generating L-polytopes. So, we shall always suppose in the paper that *the considered L-polytopes are generating.* All the examples of L-polytopes we shall deal with in the paper are, in fact, basic; we conjecture that, indeed, every generating L-polytope is basic. This property of being basic plays a crucial role, in particular, for establishing good lower bounds on the rank of an L-polytope, as we shall see in section 6.

Equivalently, one can define L-polytopes in a more intrinsic way as follows. Let P be a full dimensional polytope in $\mathbb{R}^k$ and let $V(P)$ be its set of vertices. Up to translation, we can suppose that the origin is a vertex of P. Then, P is an L-polytope if and only if the following conditions hold:

P is inscribed on some sphere S, i.e. there exists a vector $c \in \mathbb{R}^k$ such that $\|x - c\| = \|c\|$ for all $x \in V(P)$ (2.4)

The $\mathbb{Z}$-module $L(P) = \left\{ \sum_{v \in V(P)} z_v v : z_v \in \mathbb{Z} \right\}$ is a lattice (2.5)

For all $v \in L(P)$, $\|v - c\| \geq \|c\|$ holds. (2.6)

Note that, in general, the lattice $L(P)$, defined by (2.5), is a sublattice of the lattice L in which P is an L-polytope and both lattices coincide precisely when P is a generating L-polytope in L.

Note also that, if P is an L-polytope in a lattice L and if we choose as origin the center of the sphere circumscribed to P, denoting its radius by r, then the lattice $L(P)$ generated by the vertices of P is given by:

$$L(P) = \left\{ \sum_{v \in V(P)} z_v v : \sum_{v \in V(P)} z_v = 1, \ z \in \mathbb{Z}^{V(P)} \right\}$$

Also, every point v of L satisfies: $v \cdot v \geq r^2$ with equality if and only if v is a vertex of P.

Let us recall the connection between L-polytopes and Voronoi polytopes ([18]). If L is a lattice in $\mathbb{R}^k$ and $u_0 \in L$, the *Voronoi polytope* at u_0 is the set $P_v(u_0)$ consisting of all points $x \in \mathbb{R}^k$ that are at least as close to u_0 than to any other lattice point. Then, the vertices of the Voronoi polytope $P_v(u_0)$ are precisely the centers of the L-polytopes in L that contain u_0.

Let P, P' be two L-polytopes. One says that they have the same *type* if they are *affinely equivalent*, i.e. there exists an affine bijective transformation T such that $P' = T(P)$. One says that P, P' are *congruent* if they coincide up to orthogonal transformation and translation. Each k-dimensional lattice provides a partition of $\mathbb{R}^k$ into L-polytopes; according to Voronoi, two k-dimensional lattices are of the same *type* if the corresponding partitions are affinely equivalent. Voronoi ([18]) proved that the number of types of lattices in a given space $\mathbb{R}^k$ is finite, implying that the number of types of L-polytopes in $\mathbb{R}^k$ is finite. The latter fact is a crucial property used in [10] for proving that the hypermetric cone is polyhedral; we gave, in fact, in [10] a direct explicit proof of the finiteness of the number of types of L-polytopes in given dimension.

2.2. Hypermetric spaces and L-polytopes

Let X be a finite set, $|X| = n+1$, e.g. $X = \{0, 1, 2, \ldots, n\}$ and let d be a real valued function defined on all pairs of points of X such that $d_{ij} = d_{ji}$ for all $i, j \in X$ and $d_{ii} = 0$ for all $i \in X$. We consider d as a vector of $\mathbb{R}^{\binom{n+1}{2}}$, $d = (d_{ij})_{0 \le i < j \le n}$. Then, d is a *semi-metric* on X if d satisfies the triangle inequality:

$$d_{ij} \le d_{ik} + d_{jk} \text{ for all } i, j, k \in X \tag{2.7}$$

and then the pair (X, d) is called a *semi-metric space.* The family of all semi-metrics on X forms a cone $M(X) = M_{n+1}$, called *metric cone.* A semi-metric d is a *metric* if $d_{ij} \neq 0$ for all $i \neq j \in X$. It follows easily from the triangle inequality (2.7) that every semi-metric (X, d) can be obtained as *extension* of some metric subspace (Y, d) of (X, d) (by extension, we mean that each point of Y is replaced by some class of points, distances between points of the same class are set to zero, distances between any two points of two distinct classes take a common value defined by (Y, d)).

The semi-metric space (X, d) is called a *hypermetric* space if d satisfies all *hypermetric inequalities:*

$$\sum_{0 \le i < j \le n} z_i z_j d_{ij} \le 0 \quad \text{for} \quad z \in \mathbb{Z}^{n+1}, \sum_{i=0}^{n} z_i = 1 \tag{2.8}$$

Sometimes, we shall write the hypermetric inequality (2.8) as: $\sum_{i,j \in X} z_i z_j d_{ij} \le 0$; there is no ambiguity since $\sum_{i,j \in X} z_i z_j d_{ij} = 2 \left(\sum_{0 \le i < j \le n} z_i z_j d_{ij} \right)$ holds.

The hypermetric inequality (2.8) is said to be *trivial* if all z_i are equal to 0 except one z_i equal to 1. The family of all hypermetrics on X forms a cone $H(X) = H_{n+1}$ called the *hypermetric cone.*

Given a subset S of X, the *cut metric* d_S is defined by $(d_S)_{ij} = 1$ if $|S \cap \{i, j\}| = 1$ and $(d_S)_{ij} = 0$ otherwise. The *cut cone* $C(X) = C_{n+1}$ is the cone generated by all cut metrics. We have the inclusions: $C_{n+1} \subseteq H_{n+1} \subseteq$

$M_{n+1} \subseteq \mathbb{R}^{\binom{n+1}{2}}$; the cones C_{n+1}, M_{n+1} are clearly polyhedral and it was proved in [10] that H_{n+1} too is polyhedral.

The following connection between hypermetric spaces and L-polytopes was established in [2], [10]. Let P be an L-polytope with set of vertices $V(P)$, then the space $(V(P), d_0)$ is a hypermetric space, where d_0 is defined by (2.1). Conversely, let (X, d) be a hypermetric space. Then, there exists a lattice L_d of dimension k, $k \leq n = |X| - 1$, an L-polytope P_d in L_d and a map $\phi_d : X \to V(P_d)$ which is *generating*, i.e. $\phi_d(X)$ generates L_d, such that:

$$d_{ij} = d_0(\phi_d(i), \phi_d(j)) = \|\phi_d(i) - \phi_d(j)\|^2 \text{ for } i, j \in X \tag{2.9}$$

the triple (L_d, P_d, ϕ_d) being uniquely determined by (X, d) up to congruence.

Note that hypermetric spaces correspond indeed to generating L-polytopes. For example, a hypermetric d belongs to the interior of the hypermetric cone $H(X)$ if and only if P_d is a simplex ([10], see also section 6). A hypermetric d belongs to the cut cone $C(X)$ if and only if the associated lattice L_d can be embedded into a grid, or, equivalently, the set of vertices of P_d is contained in the set of vertices of some parallelepiped ([2]). Also, if $d = d_S$ is a cut metric, then the associated L-polytope P_d is simply the segment $I = [0, 1]$ in $\mathbb{R}$ (setting $\phi_d(i) = 1$ if $i \in S$ and $\phi_d(i) = 0$ if $i \in X - S$).

Let P be an L-polytope in a lattice L and denote by r the radius of the sphere circumscribing P. Let us choose as origin the center of P. Then, every point $v \in L$ satisfies: $\|v\|^2 = v \cdot v \geq r^2$ with equality if and only if v is a vertex of P. Also, for $u, v \in V(P)$,

$$d_0(u, v) = 2(r^2 - u \cdot v) \tag{2.10}$$

or, equivalently, $u \cdot v = r^2 - \frac{1}{2} d_0(u, v)$.

Remark 2.1. Note that the relation (2.10) explains a result of Seidel ([16]) stating that every hypermetric $d = (d_{ij})_{i,j \in X}$ is a Gram matrix in some space $\mathbb{R}^{1,k}$ (i.e. $\mathbb{R}^{k+1}$ equipped with the inner product: $(x, y) = x_0 y_0 - \sum_{i=1}^{k} x_i y_i$). Indeed, if (X, d) is a hypermetric space, set $v_i = \phi_d(i) \in \mathbb{R}^k$, $c_i = \sqrt{2}(r, v_i) \in \mathbb{R}^{k+1}$ for $i \in X$. Then, from (2.9), (2.10), we have that:

$$d_{ij} = d_0(v_i, v_j) = 2(r^2 - v_i \cdot v_j) = (c_i, c_j) \text{ in } \mathbb{R}^{1,k}.$$

2.3. Some basic results for hypermetric spaces and L-polytopes

We state here some first basic results on hypermetric spaces and L-polytopes that follow easily from their natural links.

Let us first make the easy observation that every L-polytope falls into one of two distinct classes. For this, we recall the notion of *antipodal* points on a sphere. Let S be a sphere of center c; for any point $x \in S$, the *antipode* of x on S is the point: $x^* = 2c - x$, i.e. the unique other point of S such that c is the midpoint of the segment $[x, x^*]$.

Definition 2.2. *Let P be an L-polytope . One says that:*
- *P is asymmetric if, for every $v \in V(P)$, $v^* \notin V(P)$*
- *P is centrally symmetric if, for every $v \in V(P)$, $v^* \in V(P)$.*

Lemma 2.3. *Every L-polytope is asymmetric or centrally symmetric.*

Proof. Assume that P has a vertex v for which the antipodal point v^* is also a vertex of P. Then, $v + v^* = 2c$ is a lattice point. Let u be another vertex of P, we check that $u^* = 2c - u$ is also a vertex of P, i.e. that $u^* \in S \cap L$. Indeed, $\|u^* - c\| = \|u - c\|$ and $u^* = 2c - u \in L$ because $u, 2c \in L$. ∎

Some trivial examples of centrally symmetric L-polytopes are the *hypercube* $\gamma_n = [0,1]^n$ (or any parallelepiped) and the *cross-polytope* $\beta_n = \text{conv}(\pm e_i : 1 \leq i \leq n)$ where e_i denotes the i-th unit vector in $\mathbb{R}^n$.

Next, we note the relation between the rank of the Cayley-Menger matrix of a hypermetric space (X, d) and the dimension of its associated L-polytope P_d.
Let $M(X, d)$ denote the *Cayley-Menger* matrix of the space (X, d), i.e. if $|X| = n + 1$, then $M(X, d)$ is the $(n + 2) \times (n + 2)$ matrix of the form $\begin{pmatrix} 0 & 1 \\ 1 & D \end{pmatrix}$ where 1 denotes the all one vector of size $n + 1$ and D denotes the $(n + 1) \times (n + 1)$ matrix whose (i, j)-th element is d_{ij}.

Proposition 2.4. $\dim(P_d) = \text{rank}(M(X, d)) - 2$.

Proof. Set $\phi_d(i) = v_i$ for all $i \in X$. Using relations (2.9), (2.10), we have that $d_{ij} = d_0(v_i, v_j) = 2r^2 - 2v_i \cdot v_j$ for all $i, j \in X$. Using standard operations on determinants, one sees easily that the matrix $M(X, d)$ has the same rank as the matrix $A(X,d) = \begin{pmatrix} 0 & 1 \\ 1 & V \end{pmatrix}$ where V is the $(n+1)\times(n+1)$ matrix whose (i,j)-th element is $v_i \cdot v_j$, i.e. $V = \text{Gram}(v_i : i \in X)$. Set $k = \dim P_d$; then the set $V_X = \{v_i : i \in X\}$ has affine rank $k+1$, so we can assume that the subset $\{v_0, v_1, \ldots, v_k\}$ of V_X is affinely independent. One checks easily that the submatrix $\begin{pmatrix} 0 & 1 \\ 1 & V_0 \end{pmatrix}$ of $A(X,d)$, where $V_0 = \text{Gram}(v_0, v_1, \ldots, v_k)$, is non-singular and that every other row of $A(X,d)$ is combination of those rows of $A(X,d)$ corresponding to the vectors v_i, $0 \le i \le k$. Hence, the matrix $A(X,d)$ and, thus, the matrix $M(X,d)$ has rank $k+2$. ■

Finally, we observe that the L-polytope associated with a subspace of the hypermetric space (X,d) is contained in the L-polytope P_d.

Proposition 2.5. *Let P be an L-polytope with set of vertices $V(P)$ and let X be a subset of $V(P)$. Let P_X denote the L-polytope associated with the hypermetric space (X,d_0). Then, $V(P_X) \subseteq V(P)$ with equality if and only if X is a generating subset of $V(P)$.*

Proof. Let P be an L-polytope in the lattice L, then L is generated by $V(P)$. Let L_X denote the sublattice of L generated by X and let A_X denote the affine space generated by X. Let S be the circumscribed sphere to P, so S is an empty sphere in L and set $S_X = S \cap A_X$. Then, S_X is an empty sphere in L_X such that $X \subseteq S_X \cap L_X$, while X generates L_X. Therefore, using the conditions (2.4) - (2.6) for the definition of L-polytopes, we deduce that $P_X = \text{conv}(S_X \cap L_X)$ is an L-polytope and it is the L-polytope associated with the hypermetric space (X,d_0). Therefore, $V(P_X) = S_X \cap L_X$ is indeed contained in $V(P) = S \cap L$. It is easy to see that $V(P_X) = V(P)$ if and only if X generates the lattice L. ■

In particular, by the above argument, one sees easily that every face of an L-polytope is an L-polytope. For instance, every 2-dimensional face of an L-polytope is a rectangle or a triangle with no obtuse angles. Obviously, there exists only one type of 1-dimensional L-polytope (the segment). One also has the following corollary.

Corollary 2.6. *Let (X,d) be a hypermetric space and (Y,d) be a subspace of (X,d), i.e. $Y \subseteq X$. Let $P_{X,d}$, $P_{Y,d}$ denote the L-polytopes associated, respectively, with (X,d), (Y,d). Then, $V(P_{Y,d}) \subseteq V(P_{X,d})$ holds.*

3. Rank of a hypermetric space

Let (X, d) be a hypermetric space, i.e. $d \in H(X)$, where $X = \{0, 1, \ldots, n\}$. We define the *annulator* $\mathrm{Ann}(X, d)$ of d by:

$$\mathrm{Ann}(X,d) = \left\{ z \in \mathbb{Z}^{n+1} : z \neq e_0, e_1, \ldots, e_n, \sum_{i=0}^{n} z_i = 1 \text{ and } \sum_{0 \leq i < j \leq n} z_i z_j d_{ij} = 0 \right\}$$

where $e_0, e_1, \ldots, e_n$ denote the unit vectors in $\mathbb{R}^{n+1}$.

Let $\mathcal{S}(X, d)$ denote the system of equations (in the variable d'):

$$\sum_{0 \leq i < j \leq n} z_i z_j d'_{ij} = 0 \quad \text{for } z \in \mathrm{Ann}(X, d) \tag{3.1}$$

Hence, $\mathcal{S}(X, d)$ is the system of the hypermetric inequalities that are satisfied at equality by d. For $z \in \mathbb{Z}^{n+1}$, let $H(z)$ denote the hyperplane of $\mathbb{R}^{\binom{n+1}{2}}$ defined by : $\sum_{0 \leq i < j \leq n} z_i z_j d'_{ij} = 0$. Then, the smallest face $F(d)$ of $H(X)$ that contains d is given by: $F(d) = H(X) \cap \bigcap_{z \in \mathrm{Ann}(X,d)} H(z)$. Note that the solution set of the system $\mathcal{S}(X, d)$ is precisely $\bigcap_{z \in \mathrm{Ann}(X,d)} H(z)$ and thus the dimension of the face $F(d)$ of $H(X)$ is equal to the rank of the solution set of $\mathcal{S}(X, d)$.

Definition 3.1. *Let (X, d) be a hypermetric space. Its rank $r(X, d)$ is the dimension of the smallest face $F(d)$ of $H(X)$ containing d, i.e. $r(X, d)$ is the rank of the solution set of the system $\mathcal{S}(X, d)$.*

Let $Z(X, d)$ denote the matrix for the system $\mathcal{S}(X, d)$, i.e. $Z(X, d)$ is the $|\mathrm{Ann}(X, d)| \times \binom{n+1}{2}$ matrix whose rows are the vectors $(z_i z_j)_{0 \leq i < j \leq n}$ for $z \in \mathrm{Ann}(d)$. Then, one obtains that:

$$r(X, d) = \binom{n+1}{2} - \mathrm{rank}(Z(X, d)) \tag{3.2}$$

In other words, all coordinates of a solution of $\mathcal{S}(X, d)$ can be expressed in terms of $r(X, d)$ ones; so $r(X, d)$ can be seen as the degree of freedom enjoyed by the solutions of $\mathcal{S}(X,d)$. For instance, if d is an interior point of $H(X)$, then $F(d) = H(X)$ and $r(X,d) = \binom{n+1}{2}$. If d lies on an extreme ray of $H(X)$, then $F(d) = \{\lambda d : \lambda \geq 0\}$ and $r(X,d) = 1$.

We shall now see that the rank of a hypermetric space (X,d) is, in fact, an invariant of its associated L-polytope; we show that $r(X,d) = r(\phi_d(X), d_0)$ holds. Following the notation of section 2.2, let P_d denote the L-polytope and let $\phi_d : X \to V(P_d)$ denote the generating map associated with (X,d). Set $V_X = \phi_d(X)$. For $u \in V_X$, set $X_u = \{i \in X : \phi_d(i) = u\}$; then, $X = \bigcup_{u \in V_X} X_u$ is a partition of X. Choose a point i_u in each class X_u. From relation (2.9) and the triangle inequalities (2.7), we deduce that:

$$d_{ij} = 0 \text{ for } i,j \in X_u \text{ and } d_{ij} = d_{i_u i_v} \text{ for } i \in X_u, j \in X_v \tag{3.3}$$

This observation implies that ϕ_d maps the system $\mathcal{S}(X,d)$ onto the system $\mathcal{S}(V_X, d_0)$. Indeed, let $\sum_{0 \leq i < j \leq n} z_i z_j d_{ij} = 0$ be a hypermetric equality satisfied by d, with $z \in \mathbb{Z}^{n+1}, \sum_{i=0}^{n} z_i = 1$. Set $y_u = \sum_{i \in X_u} z_i$ for $u \in V_X$; then $\sum_{u \in V_X} y_u = \sum_{i \in X} z_i = 1$. Using relation (2.9), we deduce that:

$$0 = \sum_{i,j \in X} z_i z_j d_{ij} = \sum_{u,v \in V_X} y_u y_v d_0(u,v) \tag{3.4}$$

and, thus, $\sum_{u,v \in X} y_u y_v d(u,v) = 0$ is an equation of the system $\mathcal{S}(V_X, d_0)$. In fact, as we prove below, the solution sets of both systems $\mathcal{S}(X,d)$ and $\mathcal{S}(V_X, d_0)$ have the same rank.

Proposition 3.2. *With the above notation,* $r(X,d) = r(V_X, d_0)$.

Proof. We show that the solution sets of both systems $\mathcal{S}(X,d)$ and $\mathcal{S}(V_X, d_0)$ have the same rank. Note that, if ϕ_d is a one-to-one map, then all classes X_u are, in fact, singletons and so the two systems $\mathcal{S}(X,d)$ and $\mathcal{S}(V_X, d_0)$ clearly coincide. Else, since d satisfies (3.3), we deduce that the equations (in the variable d):

$$d_{ii_u} + d_{ik} - d_{ki_u} = 0 \text{ and } d_{ii_u} + d_{ki_u} - d_{ik} = 0$$

for $i \in X_u, k \in X - X_u$ and $u \in V_X$, belong to the system $\mathcal{S}(X,d)$. Therefore, any solution d' of $\mathcal{S}(X,d)$ satisfies (3.3) too. So, if we substitute in the equations of $\mathcal{S}(X,d)$ using relations (3.3) and (3.4), we obtain the system $\mathcal{S}(V_X, d_0)$. Therefore, both systems $\mathcal{S}(X,d)$ and $\mathcal{S}(V_X, d_0)$ have the same rank. ■

Proposition 3.2 allows us to consider only the case when the map ϕ_d is one-to-one, i.e. to work with subspaces of the hypermetric space $(V(P), d_0)$, where P is some L-polytope or, in other words, to work with metrics rather than semi-metrics.

4. Rank of an L-polytope

Let P be a (generating) L-polytope in $\mathbb{R}^k$ and let $V(P)$ denote its set of vertices. Then, the space $(V(P), d_0)$ is a hypermetric space and, then, we can define the *rank* $r(P)$ of P as follows:

Definition 4.1. *The rank of an L-polytope P is defined by: $r(P) = r(V(P), d_0)$.*

Recall that a subset V of $V(P)$ is generating if V generates $V(P)$ and V is basic if V is generating and affinely independent. The main result of this section is that, for any generating subset V of $V(P)$, $r(P) = r(V, d_0)$ holds and thus, in particular, $r(P) = r(B, d_0)$ for any basic subset of $V(P)$. In order to prove this result, we need to give a more detailed description of the system of hypermetric equations $\mathcal{S}(V, d_0)$.

In this section, we choose as origin the center of our L-polytope P. Let r denote the radius of the circumscribed sphere to P. We recall (see section 2.1 and (2.10)) that $L = \left\{ \sum_{v \in V(P)} z_v v : \ z \in \mathbb{Z}^{V(P)} \text{ and } \sum_{v \in V(P)} z_v = 1 \right\}$, $v \cdot v \geq r^2$ for all $v \in L$ with equality if and only if $v \in V(P)$ and

$$u \cdot v = r^2 - \frac{1}{2} d_0(u,v) \text{ for } u,v \in V(P). \tag{4.1}$$

Let V be a generating subset of $V(P)$. Recall that the system $\mathcal{S}(V, d_0)$ consists of all the hypermetric inequalities of the cone $H(V)$ that are satisfied at equality by d_0. In fact, the equations of the system $\mathcal{S}(V, d_0)$ are in one-to-one correspondence with the affine representations of the points of $V(P)$ in terms of the generating set V. Indeed, let $y \in \mathrm{Ann}(V, d_0)$, so $\sum_{v \in V} y_v = 1$ and $\sum_{u,v \in V} y_u y_v d_0(u, v) = 0$. Set $w = \sum_{v \in V} y_v u$, then $w \in V(P)$, because $w \in L$ and, using (4.1) and the following relation (4.2), $w \cdot w = r^2$.

$$w \cdot w = \sum_{u,v \in V} y_u y_v u \cdot v = r^2 - \frac{1}{2} \sum_{u,v \in V} y_u y_v d_0(u, v) \tag{4.2}$$

Conversely, take $w \in V(P)$ and let $w = \sum_{v \in V} y_v v$ be an affine representation of w with $\sum_{v \in V} y_v = 1$, $y \in \mathbb{Z}^V$. Then, relation (4.2) implies that $\sum_{u,v \in V} y_u y_v d_0(u, v) = 0$, i.e. we obtain an equation of $\mathcal{S}(V, d_0)$. Note that, in general, any $w \in V(P)$ may have infinitely many affine representations in V. We have shown the following Lemma 4.2.

Lemma 4.2. *Let P be an L-polytope with set of vertices $V(P)$. Let V be a generating subset of $V(P)$. Then, there is one-to-one correspondence between:*

(i) the affine representations: $w = \sum_{v \in V} y_v v$ with $y \in \mathbb{Z}^V$, $\sum_{v \in V} y_v = 1$ of the points $w \in V(P)$ and

(ii) the equations: $\sum_{u,v \in V} y_u y_v d_0(u, v) = 0$ with $y \in \mathbb{Z}^V$, $\sum_{v \in V} y_v = 1$, of the system $\mathcal{S}(V, d_0)$.

Lemma 4.3. *Let P be an L-polytope with set of vertices $V(P)$ and let V be a generating subset of $V(P)$. Let $z \in \mathbb{Z}^V$ such that $\sum_{v \in V} z_v = 0$ and $\sum_{v \in V} z_v v = 0$. Then, the equations:*

$$\sum_{v \in V} z_v d(v, w) = 0 \text{ for } w \in V \tag{4.3}$$

$$\sum_{u,v \in V} z_u z_v d(u, v) = 0 \tag{4.4}$$

are implied by the equations of the system $\mathcal{S}(V, d_0)$.

Proof. Take $w \in V$. Set $y_w = z_w + 1, y_v = z_v$ for $v \in V$, $v \neq w$. Thus, $\sum_{v \in V} y_v = 1$ and $w = \sum_{v \in V} y_v v$. By Lemma 4.2(i), the hypermetric equation: $\sum_{u,v \in V} y_u y_v d(u,v) = 0$ belongs to the system $\mathcal{S}(V, d_0)$. If we substitute y by z, we obtain that:

$$0 = \sum_{u \in V, u \neq w} z_u \left(\sum_{v \in V, v \neq w} z_v d(u,v) + (z_w + 1) d(u,w) \right)$$
$$+ (z_w + 1) \left(\sum_{v \in V, v \neq w} z_v d(v,w) \right)$$

and thus:

$$\sum_{u,v \in V} z_u z_v d(u,v) + 2 \sum_{u \in V, u \neq w} z_u d(u,w) = 0. \tag{$*$}$$

If we multiply relation $(*)$ by z_w and sum over $w \in V$, we obtain that: $2 \sum_{u,w \in V} z_u z_w d(u,w) = 0$ and, therefore, the equation (4.4) indeed follows from $\mathcal{S}(V, d_0)$. Now, relation $(*)$ implies that the equation (4.3) too follows from $\mathcal{S}(V, d_0)$. ■

For any $w \in V(P) - V$, let $w = \sum_{v \in V} y_v^w v$ denote an affine representation of w in V, where $y^w \in \mathbb{Z}^V, \sum_{v \in V} y_v^w = 1$. Let $\mathcal{S}^*(V, d_0)$ denote the system consisting of the equations (4.3),(4.4), together with the equations:

$$\sum_{u,v \in V} y_u^w y_v^w d(u,v) = 0 \tag{4.5}$$

In other words, $\mathcal{S}^*(V, d_0)$ consists of the hypermetric equations defined by some fixed affine representations of the points of $V(P) - V$ together with the equations (4.3),(4.4) associated with the affine dependencies on V.

Lemma 4.4. *The systems $\mathcal{S}(V, d_0)$ and $\mathcal{S}^*(V, d_0)$ have both the same solutions.*

Proof. From Lemma 4.3, we know that the system $\mathcal{S}^*(V, d_0)$ is implied by the system $\mathcal{S}(V, d_0)$. Conversely, we check that every equation of $\mathcal{S}(V, d_0)$ is implied by $\mathcal{S}^*(V, d_0)$. For $w \in V(P) - V$, let $w = \sum_{v \in V} y_v v$ be another affine decomposition of w in V; set $z_v = y_v^w - y_v$, thus $\sum_{v \in V} z_v = \sum_{v \in V} z_v v = 0$. Using (4.4), we deduce that $\sum_{u,v \in V} z_u z_v d(u, v) = 0$, i.e.

$$\sum_{u,v \in V} y_u^w y_v^w d(u,v) - 2 \sum_{u,v \in V} y_u^w y_v d(u,v) + \sum_{u,v \in V} y_u y_v d(u,v) = 0 \qquad (*)$$

The first term in the left hand side of (*) is equal to 0 (using (4.5)). On the other hand, from (4.3), we deduce that: $\sum_{v \in V} y_v^w d(u,v) = \sum_{v \in V} y_v d(u,v)$ and, thus, the second term in (*) is equal to: $-2 \sum_{u,v \in V} y_u y_v d(u,v)$. We hence deduce from (*) that $\sum_{u,v \in V} y_u y_v d(u,v) = 0$. ■

Remark 4.5. *In particular, if P is a basic L-polytope, i.e. if there exists an affine basis B of L such that $B \subseteq V(P)$, then (in view of next Theorem 4.6) $r(P) = r(B, d_0)$ is the rank of the solution set to the system $\mathcal{S}^*(B, d_0)$ which consists of $|V(P)| - k - 1$ equations in $\binom{k+1}{2}$ variables. Indeed, there are no affine dependencies on B, i.e. no equations (4.3), (4.4). For $w \in V(P) - B$, let $w = \sum_{v \in B} y_v^w v$ denote the affine decomposition of w in B, with $y^w \in \mathbb{Z}^B$, $\sum_{v \in B} y_v^w = 1$. Then, let $h(w) := \sum_{u,v \in B} y_u^w y_v^w d(u,v) = 0$ denote the corresponding hypermetric equation of $\mathcal{S}^*(B, d_0)$. Hence, the system $\mathcal{S}^*(B, d_0)$ consists of the $|V(P)| - k - 1$ equations $h(w) = 0$ in the $\binom{k+1}{2}$ variables $d(u, v)$ for $u, v \in B$.*

Theorem 4.6. *Let P be an L-polytope with set of vertices $V(P)$ and let V be a generating subset of $V(P)$. Then, $r(V, d_0) = r(P)$ holds.*

Proof. In order to prove that $r(V, d_0) = r(V(P), d_0)$, we show that the sets of solutions of both systems $\mathcal{S}(V, d_0)$ and $\mathcal{S}(V(P), d_0)$ have the same rank. The system $\mathcal{S}(V, d_0)$ is contained in the system $\mathcal{S}(V(P), d_0)$; in fact, $\mathcal{S}(V, d_0)$ consists of all the equations of $\mathcal{S}(V(P), d_0)$ which have zero coefficient in the variables $d(u, v)$ for $u, v \in V(P) - V$ or $u \in V, v \in V(P) - V$. We show in the following Lemma 4.7 that the latter variables can be expressed in terms of the variables $d(u,v)$ for $u,v \in V$. Therefore, the solution sets to both systems $\mathcal{S}(V,d_0)$, $\mathcal{S}(V(P),d_0)$ have indeed the same rank. ■

Given $v,w \in V(P)-V$, let $v = \sum_{u\in V} y_u^v u$, with $y^v \in \mathbb{Z}^V$, $\sum_{u\in V} y_u^v = 1$ and $w = \sum_{u\in V} y_u^w u$, with $y^w \in \mathbb{Z}^V$, $\sum_{u\in V} y_u^w = 1$ denote the affine decompositions of v,w in the generating set V considered in relation (4.5).

Lemma 4.7. *Let d be an arbitrary solution to the system $\mathcal{S}(V(P),d_0)$. Then, d satisfies the following equations:*

$$d(v,u') = \sum_{u\in V} y_u^v d(u,u') \quad \text{for } u' \in V, v \in V(P)-V \tag{4.6}$$

$$d(v,w) = \sum_{u,x\in V} y_u^v y_x^w d(u,x) \quad \text{for } v,w \in V(P)-V \tag{4.7}$$

Proof. One can check easily that (4.6), (4.7) hold for the solution $d = d_0$. We take now an arbitrary solution d of $\mathcal{S}(V(P),d_0)$ and we show that d satisfies (4.6), (4.7). Let us define $z,z' \in \mathbb{Z}^{V(P)}$ by $z_v = -1, z_u = y_u^v$ for $u \in V$ and $z_x = 0$ for $x \in V(P) - V \cup \{v\}, z'_w = -1, z'_u = y_u^w$ for $u \in V$ and $z'_x = 0$ for $x \in V(P) - V \cup \{w\}$. Then, $\sum_{v\in V(P)} z_v = \sum_{v\in V(P)} z'_v = 0$ and $\sum_{v\in V(P)} z_v v = \sum_{v\in V(P)} z'_v v = 0$. We can apply Lemma 4.3. First, we deduce that d satisfies the equation (4.3) , i.e. $\sum_{v\in V(P)} z_v d(v,u') = 0$ for any $u' \in V$. Substituting z by its value, we obtain that: $\sum_{u\in V} y_u^v d(u,u') - d(v,u') = 0$ and, therefore, relation (4.6) holds.

Now, d satisfies the equation (4.4) for the choices of $z, z', z+z' \in \mathbb{Z}^{V(P)}$. Thus, we have the equalities: $\sum_{u,x\in V(P)} z_u z_x d(u,x) = 0$, $\sum_{u,x\in V(P)} z'_u z'_x d(u,x) = 0$ and $\sum_{u,x\in V(P)} (z_u + z'_u)(z_x + z'_x) d(u,x) = 0$. If we develop the third equation and use the first two equations above, we deduce that: $\sum_{u,x\in V(P)} z_u z'_x d(u,x) = 0$. We first substitute z' by its value in the latter equation; thus, we have that: $\sum_{u\in V(P)} z_u \left(\sum_{x\in V} y_x^w d(u,x) - d(u,w)\right) = 0$, which, together with: $\sum_{u\in V(P)} z_u d(u,w) = 0$ from (4.3), implies that: $\sum_{u\in V(P), x\in V} z_u y_x^w d(u,x) = 0$. We now substitute the value of z and obtain that:

$$\sum_{u,x\in V} y_u^v y_x^w d(u,x) - \sum_{x\in V} y_x^w d(v,x) = 0.$$

The second term is equal to: $\sum_{x\in V} y_x^w d(v,x) = \sum_{x\in V} z'_x d(v,x) + d(v,w) = d(v,w)$, using again (4.3). Hence, d satisfies relation (4.7). ■

As an application of the above results, we show that the points of the hypermetric cone that lie on the interior of the same face correspond to affinely equivalent L-polytopes. This fact was already mentioned implicitely in [10]. Let P be an L-polytope in $\mathbb{R}^k$ and T be an affine bijective map of $\mathbb{R}^k$. If $T(P)$ is again an L-polytope, then the metric d_T defined on $V(P)$ by:

$$d_T(u,v) = ||T(u) - T(v)||^2 \text{ for } u,v \in V(P) \tag{4.8}$$

is again a hypermetric on $V(P)$.

Proposition 4.8. *Let P be an L-polytope with set of vertices $V(P)$ and let $F(d_0)$ denote the smallest face of the hypermetric cone $H(V(P))$ defined on $V(P)$ that contains the hypermetric d_0, where $d_0(u,v) = ||u - v||^2$ for $u,v \in V(P)$.*

(i) if T is an affine bijective map such that $T(P)$ is an L-polytope, then d_T lies in the interior of the face $F(d_0)$.

(ii) if $d \in H(V(P))$ lies in the interior of the face $F(d_0)$, then the L-polytope associated with d is affinely equivalent to P.

Proof. We first prove (i). It suffices to check that $d_T \in F(d_0)$ (the same reasoning would give that d_0 belongs to the smallest face of $H(V(P))$ containing d_T and hence d_T lies in the interior of $F(d_0)$). We deduce from Lemma 4.4 that $F(d_0)$ consists of the points $d \in H(V(P))$ that are solutions to the system $S^*(V(P), d_0)$, i.e. satisfy the equations (4.3),(4.4) (no equations (4.5) if $V = V(P)$). Take $z \in \mathbb{Z}^{V(P)}$ such that $\sum_{v\in V(P)} z_v = \sum_{v\in V(P)} z_v v = 0$.

Therefore, $\sum_{v \in V(P)} z_v T(v) = 0$ holds. But, d_T clearly satisfies the equations $\sum_{v \in V(P)} z_v d_T(v, w) = 0$ for $w \in V(P)$ and $\sum_{u,v \in V(P)} z_u z_v d_T(u, v) = 0$ of the system $\mathcal{S}^*(V(P), d_0)$. Thus, d_T belongs to the face $F(d_0)$.

We now prove (ii). Take $d \in H(V(P))$ such that d lies on the interior of the face $F(d_0)$. Therefore, $d(u, v) \neq 0$ for all $u, v \in V(P)$. Let P' denote the L-polytope associated with d; then, there exists a bijective map T from $V(P)$ to $V(P')$ such that $d(u, v) = ||T(u) - T(v)||^2$ for $u, v \in V(P)$. In order to show that T can be extended to an affine map of the whole space, we need only check that T preserves the affine dependencies on V(P), i.e. if, for $z \in \mathbb{Z}^{V(P)}$, $\sum_{v \in V(P)} z_v = \sum_{v \in V(P)} z_v v = 0$ holds, then $\sum_{v \in V(P)} z_v T(v) = 0$ holds too (then, by continuity, the same property will hold for affine dependencies with real coefficients). But, then, since $d \in F(d_0)$, d satisfies: $\sum_{u,v \in V(P)} z_u z_v d(u, v) = 0$ and thus, $0 = \sum_{u,v \in V(P)} z_u z_v ||T(u) - T(v)||^2$ which, using the fact that the vertices of P' lie on a sphere, implies that $0 = || \sum_{u \in V(P)} z_u T(u)||^2$, i.e. $0 = \sum_{u \in V(P)} z_u T(u)$. ■

Corollary 4.9. *Let P be an L-polytope in $\mathbb{R}^k$ and let $F(d_0)$ denote the smallest face of the cone $H(V(P))$ that contains d_0. Then, $F(d_0)$ is an extreme ray of $H(V(P))$ if and only if the only (up to congruence) affine bijective transformations T of $\mathbb{R}^k$ for which $T(P)$ is an L-polytope are the homotheties.*

Proof. Assume first that $F(d_0)$ is an extreme ray of $H(V(P))$. Let T be an affine transformation such that $T(P)$ is an L-polytope. By Proposition 4.8(i), the point d_T defined by (4.8) lies on $F(d_0)$ and, thus, $d_T = \lambda^2 d_0$ for some $\lambda \geq 0$. Therefore, $d_T(u, v) = \|T(u) - T(v)\|^2 = \|\lambda u - \lambda v\|^2$ for all $u, v \in V(P)$. It is not difficult to see that $\lambda^{-1}T$ is a congruence (i.e. an orthogonal transformation composed with some translation).

Conversely, assume that the only affine bijective transformations T for which $T(P)$ is an L-polytope are the homotheties. Let d be a point lying in the interior of the face $F(d_0)$. By Proposition 4.8(ii), the L-polytope associated with d is affinely equivalent to P and thus λP is the L-polytope associated with d for some $\lambda \geq 0$. Therefore, $d(u, v) = ||\lambda u - \lambda v||^2 =$

$\lambda^2 d_0(u,v)$ for $u, v \in V(P)$, implying that $d = \lambda^2 d_0$ and, thus, $F(d_0)$ is an extreme ray of $H(V(P))$. ■

5. Reducible L-polytopes

In this section, we give a formula for the rank of an L-polytope P which is the direct product of two other L-polytopes; we show that $r(P) = r(P_1) + r(P_2)$ if $P = P_1 \times P_2$.

Let L_i be a lattice in $\mathbb{R}^{k_i}$, P_i be an L-polytope in L_i whose circumscribed sphere has radius r_i and is centered in the origin, for $i = 1,2$. One checks easily that the set $L = L_1 \times L_2 = \{(x_1, x_2) : x_1 \in L_1, x_2 \in L_2\}$ is a lattice in $\mathbb{R}^k = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, $k = k_1 + k_2$. Note that $(x_1, x_2) \cdot (y_1, y_2) = x_1 \cdot y_1 + x_2 \cdot y_2$ for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^k$. Also, $P = P_1 \times P_2 = \{(x_1, x_2) : x_1 \in P_1, x_2 \in P_2\}$ is is an L-polytope in the lattice L whose circumscribed sphere has radius $r = \sqrt{r_1^2 + r_2^2}$ and is centered in the origin. So, the direct product of L-polytopes is again an L-polytope. Call an L-polytope *reducible* if it is the direct product of two other L-polytopes and *irreducible* otherwise. Observe that reducible L-polytopes arise in reducible lattices. A lattice L in $\mathbb{R}^k$ is *reducible* if $\mathbb{R}^k$ is the orthogonal sum of two subspaces R_1, R_2 such that the projection $L_i = p_i(L)$ on R_i is a non trivial (distinct from $\{0\}$) sublattice of L, for $i = 1,2$. It was observed in [10] that, if P is an L-polytope in L, then $P_i = p_i(P)$ is an L-polytope in L_i and so, P is reducible, since $P = P_1 \times P_2$.

Theorem 5.1. *Let P_i be an L-polytope in $\mathbb{R}^{k_i}$, for $i = 1,2$. Then, $P_1 \times P_2$ is an L-polytope in $\mathbb{R}^{k_1+k_2}$ whose rank is given by: $r(P_1 \times P_2) = r(P_1) + r(P_2)$.*

Proof. Let V_i denote the set of vertices of P_i , for $i = 1,2$ and let $V = V_1 \times V_2$ denote the set of vertices of $P = P_1 \times P_2$. From Lemma 4.4, $r(P_i) = r(V_i, d_0)$ is the rank of the solution set to the system $\mathcal{S}^*(V_i, d_0)$, for $i = 1,2$, and $r(P) = r(V, d_0)$ is the rank of the solution set to the system $\mathcal{S}^*(V, d_0)$. The systems $\mathcal{S}(V_i, d_0), \mathcal{S}(V, d_0)$ consist only of the equations (4.3),(4.4) associated with the affine dependencies on V_i, V, respectively. Let b_i be a given point of V_i, for $i = 1,2$. Clearly, if $\sum_{v_1 \in V_1} z_{v_1} v_1 = 0$ with $z \in \mathbb{Z}^{V_1}$, $\sum_{v_1 \in V_1} z_{v_1} = 0$, is an affine dependency on V_1, then $\sum_{v_1 \in V_1} z_{v_1}(v_1, b_2) = 0$ is an affine dependency on V. Let $\mathcal{S}_1$ denote the subsystem of $\mathcal{S}^*(V, d_0)$ formed by the equations involving only the variables $d((u_1, b_2), (v_1, b_2))$ for $u_1, v_1 \in V_1$, i.e. $\mathcal{S}_1$ consists only of the equations: $\sum_{v_1 \in V_1} z_{v_1} d((v_1, b_2), (x_1, b_2)) = 0$ for $x_1 \in V_1$ and $\sum_{u_1, v_1 \in V_1} z_{u_1} z_{v_1} d((u_1, b_2), (v_1, b_2)) = 0$, where $\sum_{v_1 \in V_1} z_{v_1} = \sum_{v_1 \in V_1} z_{v_1} v_1 = 0$. Therefore, the systems $\mathcal{S}_1$ and $\mathcal{S}^*(V_1, d_0)$ are in one-to-one correspondence and their solution sets have the same rank, equal to $r(P_1)$. Similarly, the subsystem $\mathcal{S}_2$ of $\mathcal{S}^*(V, d_0)$ consisting of the equations involving only the variables $d((b_1, u_2), (b_1, v_2))$ for $u_2, v_2 \in V_2$ has the same rank, namely, equal to $r(P_2)$, as the system $\mathcal{S}^*(V_2, d_0)$.

We now show that the remaining variables $d((u_1, u_2), (v_1, v_2))$, for $(u_1, u_2), (v_1, v_2) \in V - (V_1 \times \{b_2\} \cup \{b_1\} \times V_2)$, can be expressed in terms of the variables $d((u_1, b_2), (v_1, b_2)), d((b_1, u_2), (b_1, v_2))$ for $u_1, v_1 \in V_1, u_2, v_2 \in V_2$. Namely, we show that the following relation (5.1) follows from the system $\mathcal{S}^*(V, d_0)$.

$$\begin{aligned} d((u_1, u_2), (v_1, v_2)) &= d((b_1, u_2), (b_1, v_2)) + d((u_1, b_2), (v_1, b_2)) \\ &\quad \text{for } u_1, v_1 \in V_1, u_2, v_2 \in V_2 \end{aligned} \tag{5.1}$$

Then, it will follow that $r(P)$ is equal to the rank of the solution set to the subsystem $\mathcal{S}_1 \cup \mathcal{S}_2$, i.e. to $r(P_1) + r(P_2)$, since the systems $\mathcal{S}_1, \mathcal{S}_2$ have disjoint sets of variables.

Relation (5.1) clearly holds for the point d_0. We now check that it holds for any solution d to the system $\mathcal{S}^*(V, d_0)$ (or $\mathcal{S}(V, d_0)$). For this, we can use the hypermetric equations of $\mathcal{S}(V, d_0)$ or the equations (4.3),(4.4) associated with the affine dependencies on V. Consider the identity:

$$(u_1, u_2) = (v_1, v_2) + (b_1, u_2) + (u_1, b_2) - (v_1, b_2) - (b_1, v_2) \tag{5.2}$$

From (5.2), we deduce the following hypermetric equation of $\mathcal{S}(V, d_0)$:

$$\begin{aligned} &d((b_1, v_2), (v_1, b_2)) + d((v_1, v_2), (b_1, u_2)) + d((v_1, v_2), (u_1, b_2)) \\ &+ d((b_1, u_2), (u_1, b_2)) - \sum_{\substack{(x_1, x_2) = (v_1, v_2), (b_1, u_2), (u_1, b_2) \\ (y_1, y_2) = (b_1, v_2), (v_1, b_2)}} d((x_1, x_2), (y_1, y_2)) = 0 \end{aligned} \tag{5.3}$$

The identity (5.2) also yields an affine dependency on V from which we deduce the following equation of $\mathcal{S}^*(V, d_0)$:

$$\begin{aligned} & d((v_1,v_2),(b_1,u_2)) + d((v_1,v_2),(u_1,b_2)) + \\ & + d((b_1,u_2),(u_1,b_2)) + d((b_1,v_2),(v_1,b_2)) + \\ & + d((u_1,u_2),(b_1,v_2)) + d((u_1,u_2),(v_1,b_2)) \\ & - \sum_{\substack{(x_1,x_2)=(v_1,v_2),(b_1,u_2),(u_1,b_2) \\ (y_1,y_2)=(u_1,u_2),(b_1,v_2),(v_1,b_2)}} d((x_1,x_2),(y_1,y_2)) = 0 \end{aligned} \tag{5.4}$$

By subtracting (5.3) from (5.4), we deduce that:

$$\begin{aligned} 0 = & \, d((u_1,u_2),(b_1,v_2)) + d((u_1,u_2),(v_1,b_2)) - d((u_1,u_2),(b_1,u_2)) \\ & - d((u_1,u_2),(u_1,b_2)) - d((u_1,u_2),(v_1,v_2)) \end{aligned} \tag{5.5}$$

The next two hypermetric equations follow, respectively, from the identities: $(v_1,u_2) = (u_1,u_2) + (v_1,b_2) - (u_1,b_2)$ and $(u_1,v_2) = (u_1,u_2) + (b_1,v_2) - (b_1,u_2)$.

$$d((u_1,u_2),(u_1,b_2)) + d((u_1,b_2),(v_1,b_2)) - d((u_1,u_2),(v_1,b_2)) = 0 \tag{5.6}$$

$$d((u_1,u_2),(b_1,u_2)) + d((b_1,v_2),(b_1,u_2)) - d((u_1,u_2),(b_1,v_2)) = 0 \tag{5.7}$$

Adding (5.6),(5.7) to (5.5), we deduce the relation from (5.1). ■

For instance, let $I = [0,1]$ be a segment in $\mathbb{R}$; then, I is an L-polytope in $\mathbb{R}$ of rank $r(I) = 1$. The k-dimensional hypercube $\gamma_k = I^k$ is the direct product of k segments; therefore, $r(\gamma_k) = kr(I) = k$.

6. Bounds for the rank of basic L-polytopes

In this section, we give some bounds for the rank of a basic L-polytope. Let P be a basic L-polytope in a lattice L , i.e. there exists an affine basis B of L such that B is contained in the set of vertices $V(P)$ of P. Then, the system $\mathcal{S}(B, d_0)$ has a compact formulation, permitting to derive easily good lower and upper bounds for the rank of basic L-polytopes. Let us recall some notation. As in Remark 4.5, for every $w \in V(P) - B$, let $w = \sum_{u\in B} y^w_u u$ denote the (unique) affine decomposition of w in B, where $y^w \in \mathbb{Z}^B$, $\sum_{u\in B} y^w_u = 1$. The system $\mathcal{S}(B, d_0)$ consists of the equations $h(w) := \sum_{u,v\in B} y^w_u y^w_v d(u,v) = 0$ for $w \in V(P) - B$.

Proposition 6.1. *Let P be a basic L-polytope in $\mathbb{R}^k$ with set of vertices $V(P)$. Then,*

$$r(P) \geq \binom{k+2}{2} - |V(P)| \tag{6.1}$$

$$r(P) \leq \binom{k+1}{2} \tag{6.2}$$

Proof. From Theorem 4.6, $r(P) = r(B, d_0)$ is the rank of the solution set to the system $\mathcal{S}(B, d_0)$. Since the system $\mathcal{S}(B, d_0)$ consists of $|V(P)| - k - 1$ equations in $\binom{k+1}{2}$ variables, we deduce easily the inequalities (6.1),(6.2) (for (6.1), use relation (3.2)). ∎

We saw in section 2.3 that every L-polytope is either centrally symmetric (i.e. for every vertex v of P, its antipode v^* is a vertex of P), or asymmetric (i.e. for every vertex v of P, its antipode v^* is not a vertex of P). For centrally symmetric basic L-polytopes, we are able to refine the lower bound for the rank.

Theorem 6.2. *Let P be a basic centrally symmetric L-polytope in $\mathbb{R}^k$ with set of vertices $V(P)$. Then,*

$$r(P) \geq \binom{k+1}{2} - \frac{|V(P)|}{2} + 1. \tag{6.3}$$

We first state two Lemmas.

Lemma 6.3. *Let $w \in V(P) - B$ and $w^* \in V(P)$ be its antipode. Then, $h(w^*) = h(w) + \sum_{u \in B} y_u^w h(u^*)$.*

Proof. Let $w^* = \sum_{u \in B} z_u^w u, v^* = \sum_{u \in B} z_u^v u$ denote the affine decompositions of w^*, v^* in B with $\sum_{u \in B} z_u^w = \sum_{u \in B} z_u^{v^*} = 1$, where $v \in B$. Note that:

$$w^* = v + v^* - w = v + \sum_{u \in B} z_u^v u - \sum_{u \in B} y_u^w u.$$

Therefore, $z_u^w = z_u^v - y_u^w$ for $u \in B, u \neq v$ and $z_v^w = z_v^v - y_v^w + 1$. Therefore, we deduce that:

$$h(w^*) = h(v^*) + h(w) + 2\sum_{u \in B} z_u^v d(u,v) - 2\sum_{u \in B} y_u^w d(u,v)$$
$$- 2\sum_{u,x \in B} z_u^v y_x^w d(u,x).$$

Since $\sum_{u \in B} y_u^w = 1$, we can rewrite $h(w^*)$ as follows:

$$h(w^*) = h(w) + \sum_{u \in B} y_u^w \left(h(v^*) - 2d(u,v) + 2\sum_{x \in B} z_x^v (d(x,v) - d(x,u)) \right). \tag{6.4}$$

If $w \in B$, then $y_u^w = 0$ for $u \in B$ except $y_w^w = 1$ and $h(w)$ is identically zero. Hence, we deduce from (6.4) that:

$$h(w^*) = h(v^*) - 2d(w,v) + 2\sum_{x \in B} z_x^v \left(d(x,v) - d(x,w)\right), \; w \in B \tag{6.5}$$

If we substitute the equality (6.5) in relation (6.4), we deduce that: $h(w^*) = h(w) + \sum_{u \in B} y_u^w h(u^*)$. ■

Lemma 6.4. *For $v \in B$, $h(v^*) + \sum_{u \in B} z_u^v h(u^*) = 0$, where $v^* = \sum_{u \in B} z_u^v u$ is the affine decomposition of v^* in B.*

Proof. We apply Lemma 6.3 to $w = v^*$. Then, we obtain that: $h(v) = h(v^*) + \sum_{u \in B} z_u^v h(u^*)$. But $h(v)$ is identically zero because $v \in B$. ■

Proof of Theorem 6.2. Let B' denote the set of the antipodes of the points of B that do not belong to B. Then, $V(P) - B \cup B' = A \cup A'$, where A' is formed by the antipodes of the points of A and so $A \cap A' = \emptyset$, $|A| = |A'|$. Note that B contains at most one antipodal pair of vertices; indeed, if x, x^*, y, y^* belong to B, then $x + x^* - y - y^* = 0$ would be an affine dependency in B.

The system $\mathcal{S}(B, d_0)$ consists of the equations: $h(w) = 0$ for $w \in V(P) - B = B' \cup A \cup A'$. From Lemma 6.3, every equation $h(w) = 0$ for

$w \in A'$ follows from the equations: $h(w) = 0$ for $w \in B' \cup A$. Hence, $r(P)$ is equal to the rank of the subsystem formed by the equations: $h(w) = 0$ for $w \in B' \cup A$.

Let us first suppose that B contains exactly one antipodal pair. Then, $|B'| = k - 1$, $|A| = \frac{|V|}{2} - k$ and so, by the above observation, $\mathcal{S}(B, d_0)$ reduces to a system of $|A| + |B'| = \frac{|V|}{2} - 1$ equations. Therefore, $r(P) \geq \binom{k+1}{2} - \frac{|V|}{2} + 1$.

Suppose now that B contains no pair of antipodal points. Then, $|B'| = k+1$, $|A| = \frac{|V|}{2} - k - 1$. From Lemma 6.4 we have that one of the equations: $h(w) = 0$ for $w \in B'$ follows from the others. Therefore, the system $\mathcal{S}(B, d_0)$ reduces, in fact, to a system of $|A| + |B'| - 1 = \frac{|V|}{2} - 1$ equations. Again, this implies that $r(P) \geq \binom{k+1}{2} - \frac{|V|}{2} + 1$. ∎

We give some easy examples of L-polytopes attaining the bounds for the rank. We will see in sections 8,10,11 several other examples of L-polytopes which attain these bounds.

The k-dimensional simplex α_k has $k+1$ vertices; hence, both the lower bound and the upper bound of Proposition 6.1 are equal to $\binom{k+1}{2}$, implying that $r(\alpha_k) = \binom{k+1}{2}$. Thus, $(V(\alpha_k), d_0)$ is an interior point of the hypermetric cone H_{k+1} and, from Proposition 4.8, the L-polytope associated with any interior point of H_{k+1} is a simplex.

The k-dimensional cross-polytope β_k is a centrally symmetric L-polytope ; recall that, if $e_1, \ldots, e_k$ denote the unit vectors in $\mathbb{R}^k$, then β_k has $2k$ vertices: $\pm e_1, \ldots, \pm e_k$. So, from Theorem 6.2, $r(\beta_k) \geq \binom{k+1}{2} - k + 1$. But, in fact, β_k realizes equality in this bound. It is indeed an easy exercise to compute $r(\beta_k)$. For this, note that $B = \{e_1, e_1^*, \ldots, e_k\}$ is an affine basis (setting $e_i^* = -e_i$) and the decomposition of each non basic vertex in B is: $e_i^* = e_1 + e_1^* - e_i$ for $i = 2, \ldots, k$. Thus, the system $\mathcal{S}(B, d_0)$ consists of the $k-1$ equations: $d(e_1, e_1^*) - d(e_1, e_i) - d(e_i^*, e_i) = 0$ for $2 \leq i \leq k$, in $\binom{k+1}{2}$ variables. Hence, the rank of its solution set is equal to $r(\beta_k) = \binom{k+1}{2} - k + 1$.

Finally, we conclude with a lemma which will be useful for proving the extremality of several L-polytopes in sections 8,10 and 11.

Lemma 6.5. *Let P be a k-dimensional centrally symmetric L-polytope which is basic, i.e. there exists an affine basis B of the lattice such that $B \subseteq V(P)$, say $B = \{v_0, v_1, \ldots, v_k\}$. Let F denote the hyperplane spanned by the vectors of $B_1 = \{v_1, \ldots, v_k\}$ and set $P_1 = F \cap P$. Assume that* *(i) P_1 is an asymmetric L-polytope (of dimension $k-1$) with $V(P_1) = V(P) \cap F$ containing B_1 as an affine basis*

(ii) there exists a vector $v \in V(P) - V(P_1)$ such that $v \notin \{v_1^*, \ldots, v_k^*\}$ and $v - v_0 \notin F$.

Then, $r(P) = r(P_1)$ holds; in particular, if $r(P_1) = 1$, then $r(P) = 1$.

Proof of Lemma 6.5. From Remark 4.5, $r(P)$ (resp. $r(P_1)$) is the rank of the solution set to the system $\mathcal{S}^*(B, d_0)$ (resp. $\mathcal{S}^*(B_1, d_0)$). In order to show that $r(P) = r(P_1)$ holds, it suffices to check that the variables $d(v_0, v_i)$ for $1 \leq i \leq k$ in the system $\mathcal{S}^*(B, d_0)$ can be expressed in terms of the variables $d(v_i, v_j)$ for $1 \leq i < j \leq k$. We will check this by using the equations $h(v_i^*) = 0$ for $1 \leq i \leq k$ and $h(v) = 0$ of the system $\mathcal{S}^*(B, d_0)$.

Set $v = \sum_{0 \leq i \leq k} y_i v_i$ and $v_0^* = \sum_{0 \leq i \leq k} z_i v_i$ where y_i, z_i are integers with $\sum_{0 \leq i \leq k} y_i = \sum_{0 \leq i \leq k} z_i = 1$. Since $v \notin V(P_1)$ and $v - v_0 \notin F$, we have that $y_0 \neq 0, 1$; also, $z_0 \neq -1$, else the center $v_0 + v_0^*$ of the sphere circumscribing P would belong to F, in contradiction with the fact that P_1 is asymmetric. Using the relation (6.5) applied to the vectors v_i, v_0 of B, we deduce that:

$$h(v_i^*) = h(v_0^*) - 2d(v_i, v_0) + 2 \sum_{0 \leq j \leq k} z_j (d(v_j, v_0) - d(v_j, v_i)).$$

Set $h_i = -2 \sum_{1 \leq j \leq k} z_j d(v_j, v_i)$. Then,

$$0 = h(v_i^*) - h(v_0^*) = h_i - 2(z_0 + 1) d(v_0, v_i) + 2 \sum_{1 \leq j \leq k} z_j d(v_j, v_0).$$

Subtracting the above relations for the indices i and 1, we deduce that:

$$0 = h_i - h_1 - 2(z_0 + 1)(d(v_0, v_i) - d(v_0, v_1))$$

and, therefore, since $z_0 \neq -1$, we obtain that:

(6.6) $d(v_0, v_i) = d(v_0, v_1) + \frac{(h_i - h_1)}{2(z_0 + 1)}$ for $2 \leq i \leq k$.

We now use the equation $h(v) = 0$, i.e.

$$0 = \sum_{1 \leq i, j \leq k} y_i y_j d(v_i, v_j) + 2 \sum_{1 \leq j \leq k} y_0 y_j d(v_0, v_j).$$

Using relation (6.6), the above relation can be rewritten as:

$$0 = \sum_{1 \leq i < j \leq k} y_i y_j d(v_i, v_j) + d(v_0, v_1) y_0 (1 - y_0) + \sum_{2 \leq j \leq k} y_0 y_j \frac{(h_j - h_1)}{2(z_0 + 1)}.$$

Therefore, since $y_0 \neq 0, 1$, the variable $d(v_0, v_1)$ and thus all variables $d(v_0, v_i)$ for $1 \leq i \leq k$ can be expressed in terms of the variables $d(v_i, v_j)$ for $1 \leq i < j \leq k$. ■

7. Extreme L-polytopes

An L-polytope P is called *extreme* if $r(P) = 1$. This definition is motivated by the fact that extreme L-polytopes correspond to extreme rays of the hypermetric cone.

More precisely, if P is an extreme L-polytope, then, for every generating subset V of its set of vertices, by Theorem 4.6, $r(V, d_0) = r(P) = 1$ holds. Therefore, by definition of the rank of a hypermetric space, the hypermetric space (V, d_0) lies on an extreme ray of the hypermetric cone $H(V)$. Moreover, if (X, d) is a hypermetric space with associated generating map from X to the set of vertices of P, then, from Proposition 3.2, $r(X, d) = r(P) = 1$ and, therefore, the hypermetric space (X, d) lies on an extreme ray of the cone $H(X)$. If P is an L-polytope in $\mathbb{R}^k$, then, for any set X, $|X| \geq k + 1$, one can always find a generating map Φ from X to $V(P)$. Hence each k dimensional extreme L-polytope generates extreme rays of the hypermetric cone $H(X)$ for all X such that $|X| \geq k + 1$. Therefore, finding all extreme rays of the hypermetric cone H_n amounts to finding all extreme L-polytopes of dimension $k \leq n - 1$.

As a consequence of Corollary 4.9, extreme L-polytopes have the following nice geometric characterization. An L-polytope P is extreme if and only if the only L-polytopes that are affinely equivalent to P are its homothetic images (up to congruence).

The simplest extreme L-polytope is the segment $I = [0, 1]$ in $\mathbb{R}$; it generates all extreme rays of H_n arising from the cut metrics. It is known that, for $n \leq 6$, all extreme rays of H_n are cut metrics; this was proved in [9] for $n \leq 5$ and in [3] for $n = 6$. Therefore, there are no extreme L-polytopes of dimension k, $2 \leq k \leq 5$. For $n \geq 7$, the hypermetric cone H_n has some extreme rays that are not cut metrics. In the next section, we shall show that the two famous Gosset polytopes 2_{21} and 3_{21} that are L-polytopes in the root lattices E_6, E_7, respectively, are extreme L-polytopes. Hence, they yield extreme rays of the cones H_7 and H_8, respectively. We give a more precise description of $3_{21}, 2_{21}$ in the next section.

In fact, the segment $I \subseteq \mathbb{R}$ and the Gosset polytopes $2_{21}, 3_{21}$ are the only extreme L-polytopes arising from root lattices, i.e. integral lattices generated by vectors x with $x \cdot x = 2$. Indeed, let P be an extreme L-polytope in a root lattice L. Then, from Theorem 5.1, L must be irreducible. A well-known theorem by Witt asserts that the only irreducible root lattices are $A_n (n \geq 0)$, $D_n (n \geq 4)$ and $E_n (n = 6,7,8)$, where $A_n = \{x \in \mathbb{Z}^{n+1} : \sum_{i=1}^{n+1} x_i = 0\}$, $D_n = \{x \in \mathbb{Z}^{n+1} : \sum_{i=1}^{n} x_i \equiv 0(2)\}$, E_8 is the lattice in $\mathbb{R}^8$ spanned by D_8 and $\frac{1}{2}(e_1 + \ldots + e_8) = (\frac{1}{2}, \ldots, \frac{1}{2})$ i.e. $E_8 = \{x \in \mathbb{R}^8 :$ all $x_i \in \mathbb{Z}$ or $x_i \in \mathbb{Z} + \frac{1}{2}$ and $\sum_{i=1}^{8} x_i \equiv 0(2)\}$, E_7 is formed by all vectors in E_8 orthogonal to a given minimal (of minimal norm) vector v of E_8, i.e. $E_7 = \{x \in E_8 : x \cdot v = 0\}$ and, E_6 is formed by all vectors in E_8 orthogonal to a given A_2-sublattice of E_8, i.e. $E_6 = \{x \in E_7 : x \cdot w = 0\}$, where $\mathbb{Z}(v, w)$ is an A_2-sublattice of E_8.

The list of all types of L-polytopes in the lattices A_n, D_n, E_8, E_7, E_6, is given, in graph terms, in [17]; see also [12] for a more detailed description. In particular,

- for D_n, they are the half-cube $h\gamma_n$ (whose vertices are all $x \in \{0,1\}^n$ with $\sum x_i \equiv 0(2)$) and the cross-polytope β_n; $h\gamma_n$ corresponds to a deep hole (i.e. the radius of its circumscribed sphere is the maximum possible) while β_n corresponds to a shallow (i.e. not deep) hole in D_n.
- for E_8, they are the cross polytope β_8 and the simplex α_8 (corresponding, respectively, to deep and shallow holes).
- for E_7, they are the Gosset polytope 3_{21} and the simplex α_7 (respectively, deep and shallow holes).
- for E_6, there is only the Gosset polytope 2_{21}.

The half-cube $h\gamma_n$ is clearly not extreme, since the corresponding hypermetric belongs, in fact, to the cut cone; we saw above that the simplex α_n and the cross-polytope β_n are not extreme either.

We refer, for instance, to [4,5] for more detailed information on the above lattices.

The Gosset polytope 2_{21} is an asymmetric L-polytope having 27 vertices, while 3_{21} is a centrally symmetric L-polytope having 56 vertices.

Therefore, for the polytopes $2_{21}, 3_{21}$, the lower bounds (6.1), (6.3), respectively, are satisfied at equality. If we set $r(P) = 1$ in (6.1), (6.3), we obtain the following lower bounds on the number of vertices of an extreme L-polytope.

Theorem 7.1. *Let P be a k-dimensional basic extreme L-polytope with set of vertices $V(P)$. Then,*

$$|V(P)| \geq \frac{k(k+3)}{2} \tag{7.1}$$

$$|V(P)| \geq k(k+1) \text{ if } P \text{ is centrally symmetric} \tag{7.2}$$

There is a striking analogy between the lower bounds (7.1), (7.2) and the following known upper bounds (7.3), (7.4) for the number N_p of points in a spherical two-distance set of dimension k and the number N_ℓ of lines in an equiangular set of lines of dimension k. We refer to [8] for the bound (7.3) and to [13] for (7.4).

$$N_p \leq \frac{k(k+3)}{2} \tag{7.3}$$

$$N_\ell \leq \frac{k(k+1)}{2} \tag{7.4}$$

Note that the distances between distinct vertices of the Gosset polytope 2_{21} take only two values; hence, the set of vertices of 2_{21} is indeed a spherical two-distance set in $\mathbb{R}^6$, realizing equality in (7.3).

Since the Gosset polytope 3_{21} is centrally symmetric, we can arrange its vertices in 28 pairs of antipodal points. Each such pair determines a line going through the center of the circumscribed sphere to 3_{21}. So, we have a set of 28 lines in $\mathbb{R}^7$ which are, in fact, equiangular and they realize equality in (7.4).

Recall that equiangular sets of lines and spherical two-distance sets are in correspondence. Namely, let $\mathcal{L}$ be a set of equiangular lines of dimension $k+1$ and let $\ell_0 \in \mathcal{L}$. Choose a unit vector e_0 along ℓ_0 and, for each $\ell \in \mathcal{L}$, $\ell \neq \ell_0$, choose a unit vector e_ℓ along ℓ which forms an acute angle with e_0. Then, the set $\mathcal{P} = \{e_\ell : \ \ell \in \mathcal{L} - \{\ell_0\}\}$ is a spherical two-distance set in dimension k; indeed, if ϕ denotes the common acute angle between the lines of $\mathcal{L}$, then $\mathcal{P}$ lies on the sphere of center $\cos\phi e_0$, radius $\sin\phi$, in the hyperplane: $x \cdot e_0 = \cos\phi$. The construction can be reversed. Also, $|\mathcal{P}| = |\mathcal{L}| - 1$ and thus the two bounds (7.3), (7.4) can be deduced from one another.

The bound (7.4) was given by Gerzon who also proved that, if equality holds in (7.4), then $k+2 = 4, 5$ or $k+2 = q^2$ for some odd integer $q, q \geq 3$ (see [13]). The first case of equality in Gerzon's bound is for $q = 3, k = 7, N_\ell = 28$. So, in this case, it corresponds to the equiangular set of 28 lines related to the Gosset polytope 3_{21}. The next case of equality is $q = 5, k = 23, N_\ell = 276$. Neumaier ([15]) has shown how to construct a set of 276 equiangular lines using the Leech lattice Λ_{24}. In section 10, we shall see that some extreme centrally symmetric L-polytope of dimension 23 and with 552 vertices can be constructed from this set of lines, also that a suitable section of it is an extreme asymmetric L-polytope of dimension 22 and with 275 vertices. In other words, the situation of the Gosset polytopes $2_{21}, 3_{21}$ coming from the root lattice E_8 can be mimicked for the case of the Leech lattice. The next cases of equality in Gerzon's bound are for $q = 7, k = 47, N_\ell = 1128$ and for $q = 9, k = 79, N_\ell = 3160$; but it is not known whether such sets of equiangular lines exist in these two cases.

On the other hand, we shall see in section 11 some examples of extreme L-polytopes realizing equality in the bound (7.1) or (7.2), but not arising from some spherical 2-distance set or from some equiangular set of lines. Also, we shall have examples of extreme L-polytopes that do not realize equality in the bound (7.1) or (7.2).

Finally, let us mention a method of construction of L-polytopes in any lattice.

Let L be a lattice in $\mathbb{R}^k$ and let V be the set of minimal vectors (i.e. of minimum norm) of L. Given non collinear vectors $a, b \in \mathbb{R}^k$ and some non zero scalars α, β, we set

$$V_a = \{x \in V : \ x \cdot a = \alpha\} \text{ and } V_b = \{x \in V : \ x \cdot b = \beta\}.$$

Lemma 7.2. *If the sets V_a and $V_a \cap V_b$ are not empty, then the polytopes $P = \operatorname{conv}(V_a)$ and $P' = \operatorname{conv}(V_a \cap V_b)$ are L-polytopes.*

Proof. Set $\gamma^2 = x \cdot x$ for all $x \in V$, so V lies on the sphere S of center 0 and radius γ. Let F_a, F_b denote the hyperplanes defined by the equations: $x \cdot a = \alpha$ and $x \cdot b = \beta$, respectively. Then, $S_a = S \cap F_a$ is the $(k-1)$-dimensional sphere with center $c_a = \frac{\alpha}{a \cdot a} a$ and radius $\gamma^2 - \|c_a\|^2$ lying in the hyperplane F_a. Similarly, $S_b = S \cap F_b$ is the $(k-1)$-dimensional sphere with center $c_b = \frac{\beta}{b \cdot b} b$ and radius $\gamma^2 - \|c_b\|^2$ lying in the hyperplane F_b. Take a point $x \in V \cap F_a$, then $(x - c_a) \cdot c_a = 0$ and thus $\|x\|^2 = \gamma^2 = \|x - c_a\|^2 + \|c_a\|^2$, implying that x belongs to the sphere S_a. Thus, the polytope P is inscribed on the sphere S_a. Note that the set $L(P) = \left\{ \sum_{v \in V_a} z_v v : z_v \in \mathbb{Z}, \sum_{v \in V_a} z_v = 1 \right\}$ is a sublattice of L and thus is a lattice. Finally, for all $x \in L(P) \subseteq L \cap F_a$, $\|x - c_a\|^2 = \|x\|^2 - \|c_a\|^2 \geq \gamma^2 - \|c_a\|^2$, implying that S_a is an empty sphere in $L(P)$. Therefore, since the conditions (2.4) - (2.6) are satisfied, $P = \text{conv}(V_a)$ is indeed an L-polytope. In a similar way, we obtain that the polytope P' is inscribed on the $(k-2)$-dimensional sphere $S_{ab} = S_a \cap S_b$ which is an empty sphere in the lattice $L(P') = \left\{ \sum_{v \in V_a \cap V_b} z_v v : z_v \in \mathbb{Z}, \sum_{v \in V_a \cap V_b} z_v = 1 \right\}$ and, therefore, P' too is an L-polytope. ■

Note that this is precisely how the two Gosset polytopes $2_{21}, 3_{21}$ are constructed from the root lattice E_8, and how we construct two L-polytopes from the Leech lattice, as we shall see in sections 8 and 10.

8. The Gosset polytopes $2_{21}, 3_{21}$ are extreme

In application of our treatment of extreme L-polytopes, developed in sections 3,4,7, we show that the Gosset polytopes $2_{21}, 3_{21}$ are extreme. The proof will consist of:

- finding an affine basis B, so $|B| = 7$ for 2_{21} and $|B| = 8$ for 3_{21} (thus, both $2_{21}, 3_{21}$ are basic L-polytopes).
- using the affine decomposition of each non basic vertex in B, find the explicit description of the system $\mathcal{S}(B, d_0)$ (it consists of 27-7=20 equations for 2_{21} and $\frac{56}{2} - 1 = 27$ for 3_{21}).
- showing that the solution set to the system $\mathcal{S}(B, d_0)$ has rank 1.

So we need an explicit description of the polytope $2_{21}, 3_{21}$. We refer e.g. to [4,5] for a detailed account of the facts mentioned below. The lattice E_8

is:

$$E_8 = \{x \in \mathbb{R}^8 : \quad \text{all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \frac{1}{2} \text{ and } \sum_{i=1}^{8} x_i \equiv 0(2)\}$$

Let V_8 denote the set of minimal vectors of E_8; V_8 consists of

- the 112 vectors $(\pm 1^2, 0^6)$
- the 128 vectors $\left(\pm\frac{1}{2}^8\right)$ that have an even number of minus signs.

So, $|V_8| = 240$ and $v \cdot v = 2$ for $v \in V_8$. The set V_8 lies on the sphere S_8 of center 0, radius $\sqrt{2}$.

Let $v_0 = (1, 1, 0^6)$ be a given minimal vector. One can check that $v \cdot v_0 = 0, \pm 1$ for all $v \in V_8, v \neq \pm v_0$. Note that $\{x \in E_8 : x \cdot v_0 = 1\}$ is the lattice E_7 (up to translation). Let F_7 denote the hyperplane of equation: $x \cdot v_0 = 1$; then, $S_7 = S_8 \cap F_7$ is the 7-dimensional sphere with center $\frac{v_0}{2}$ and radius $\sqrt{\frac{3}{2}}$. Set

$$V_7 = \{x \in V_8 : x \cdot v_0 = 1\}.$$

Then, V_7 consists of

- the 12 vectors $(1, 0, \pm 1, 0^5)$
- the 12 vectors $(0, 1, \pm 1, 0^5)$
- the 32 vectors $\left(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}^6\right)$ with an even number of minus signs.

So, $|V_7| = 56$ and V_7 lies on the sphere S_7. The polytope $\text{conv}(V_7)$ is an L-polytope (recall Lemma 7.2) and it is precisely the Gosset polytope 3_{21}. Observe that the 56 points of V_7 are partitioned in 28 pairs of antipodal points (with respect to the sphere S_7, i.e. the antipode of v is $v^* = v_0 - v$). So, the polytope 3_{21} is a centrally symmetric L-polytope .

Let $w_0 = \left(\frac{1}{2}\right)^8$ be a given minimal vector of V_7. One can check that $v \cdot w_0 = 0, 1$ for all $v \in V_7, v \neq w_0$ and $v \neq w_0^*$ $\left(=\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}^6\right)\right)$. Then, the set $\{x \in E_7 : x \cdot w_0 = 1\}$ is the lattice E_6. Note also that, if v^* is the antipode of $v \in V_7$, then $v \cdot w_0 + v^* \cdot w_0 = v_0 \cdot w_0 = 1$ and, thus, $v \cdot w_0 = 1$ if and only if $v^* \cdot w_0 = 0$. Let F_6 denote the hyperplane with equation: $x \cdot w_0 = 1$; then, $S_6 = S_7 \cap F_6 = S_8 \cap F_7 \cap F_6$ is the 6-dimensional sphere with center $\frac{v_0 + w_0}{3}$ and radius $\sqrt{\frac{4}{3}}$. Set $V_6 = \{x \in V_7 : x \cdot w_0 = 1\}$, $V_6^* = \{v^* : v \in V_6\}$. Hence, $V_7 = V_6 \cup V_6^* \cup \{w_0, w_0^*\}$. The set V_6 consists of:

- the 6 vectors: $\left(1,0,1,0^5\right)$
- the 6 vectors: $\left(0,1,1,0^5\right)$
- the 15 vectors: $\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}^2,\frac{1}{2}^4\right)$

Hence, $|V_6| = 27$ and V_6 lies on the sphere S_6. The polytope conv(V_6) is the Gosset polytope 2_{21}; from Lemma 7.2, it is indeed an L-polytope and it is clearly asymmetric.

Remark 8.1.

(i) For $u, v \in V_7, v \neq u, u^*$, we have that $u \cdot v = 0, 1$ and, thus, $\left(u - \frac{v_0}{2}\right) \cdot \left(v - \frac{v_0}{2}\right) = \frac{1}{2}, -\frac{1}{2}$. Therefore, the 28 distinct lines going through $v, v^*, \frac{v_0}{2}$ for $v \in V_7$ form a 7-dimensional set of equiangular lines (with common angle arccos$(\frac{1}{3})$*).*

(ii) For $u, v \in V_6, v \neq u, u \cdot v = 0, 1$ and thus $d_0(u,v) = \|u - v\|^2 = 4$ (if $u \cdot v = 0$) or 2 (if $u \cdot v = 1$). Therefore, the 27 vertices of 2_{21} form a 6-dimensional spherical two-distance set of points.

Theorem 8.2. *The Gosset polytopes $2_{21}, 3_{21}$ are extreme.*

Remark 8.3. *The extremality of the polytopes $2_{21}, 3_{21}$ was proved by Erdahl ([12]) (in different terms).*

Proof. We denote the vectors of V_6 by: $u_i = \left(1,0,1_i,0^5\right), v_i = \left(0,1,1_i,0^5\right)$, where the first two coordinates are fixed and the second "1" stays in the $(2+i)$-th position, for $1 \leq i \leq 6$, and $u_{ij} = \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}_i,-\frac{1}{2}_j,\frac{1}{2}^4\right)$ where the two "$-\frac{1}{2}$" stay in the $(2+i)$-th and $(2+j)$-th positions for $1 \leq i < j \leq 6$. One can verify that the distances between the points of V_6 are as follows, where we set $t = 2$.

$$\begin{cases} d(u_i,u_j) & = d(v_i,v_j) = t \text{ for } i \neq j \\ d(u_i,v_j) & = \begin{cases} t & \text{if } i = j \\ 2t & \text{if } i \neq j \end{cases} \\ d(u_i,u_{kl}) & = d(v_i,u_{kl}) = \begin{cases} t & \text{if } i \notin \{k,l\} \\ 2t & \text{if } i \in \{k,l\} \end{cases} \\ d(u_{ij},u_{kl}) & = \begin{cases} t & \text{if } |\{i,j\} \cap \{k,l\}| = 1 \\ 2t & \text{if } |\{i,j\} \cap \{k,l\}| = 0 \end{cases} \end{cases} \tag{8.1}$$

Consider the following set of 7 points of V_6 :

$$B_6 = \{u_{12}, u_{24}, u_{34}, u_{35}, u_{15}, u_6, v_6\}.$$

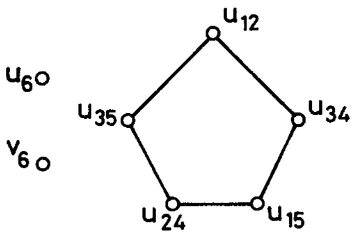

Figure 1.

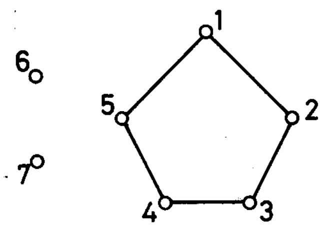

Figure 2.

The graph on B_6 whose edges are the pairs at distance $2t$ is shown in Figure 1 and, for simplicity, its nodes are renumbered as shown in Figure 2.

One can check that B_6 is an affine basis of E_6, i.e. that B_6 generates the set V_6. The affine decompositions of the non basic points of $V_6 - B_6$ in B_6 give the following system of 20 hypermetric equations in the 21 variables $d(i,j)$ for $1 \leq i < j \leq 7$ (the indices are modulo 5).

$$\begin{cases} d(i,6) + d(i+1,6) - d(i,i+1) & =0 \quad 1 \leq i \leq 5 \\ d(i,7) + d(i+1,7) - d(i,i+1) & =0 \quad 1 \leq i \leq 5 \\ d(i,i+2) + d(i,i+3) - d(i+2,i+3) & =0 \quad 1 \leq i \leq 5 \\ d(6,7) + \displaystyle\sum_{\substack{i<j \\ i,j \in \{k,k+1,k+2\}}} d(i,j) - & \\ \displaystyle\sum_{i \in \{k,k+1,k+2\}} (d(i,6) + d(i,7)) & =0 \quad 1 \leq k \leq 5 \end{cases} \tag{8.2}$$

In fact, the equations of the first, second and fourth lines corrspond to the representations of v_i, u_i and u_{k6}, respectively. The equations of the third line correspond to the representations of $u_{45}, u_{25}, u_{23}, u_{13}$ and u_{14}.

For example, the equation: $d(1,6) + d(2,6) - d(1,2) = 0$ comes from the affine decomposition of v_5 in B_6 : $v_5 = u_{12} + u_{34} - u_6$.

One can verify that the solution set to the system $\mathcal{S}(B_6, d_0)$, i.e. the system (8.2), is precisely given by (8.1) and, thus, has rank 1. Therefore, $r(2_{21}) = r(B_6, d_0) = 1$, showing that 2_{21} is extreme.

We now turn to the case of 3_{21}. Consider the set $B_7 = B_6 \cup \{w_0\}$, recall that $w_0 = \left(\frac{1}{2}\right)^8$. It is clear that B_7 is an affine basis of E_7, i.e. that B_7 generates the set V_7. Indeed, recall that $V_7 = V_6 \cup V_6^* \cup \{w_0, w_0^*\}$. We saw above that V_6 is generated by B_6. Note that $v_0 = u_{12} + u_{34} + u_{56} - w_0$; then, for $v \in V_6$, we have that $v^* = v_0 - v = u_{12} + u_{34} + u_{56} - w_0 - v$ and thus v^* is affinely decomposable in B_7. Since $w_0 \cdot v = 1$ for all $v \in B_6$, we have

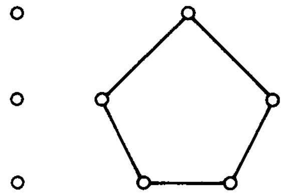

Figure 3.

that $d(w_0, v) = 2$ for $v \in B_6$ and thus the graph on B_7 whose edges are the pairs at distance $2t$ ($t = 2$) has the configuration shown in Figure 3, after denoting by 8 the point w_0.

The system $\mathcal{S}(B_7, d_0)$ consists of the system $\mathcal{S}(B_6, d_0)$ together with the 7 equations corresponding to the decomposition of v^* in B_7 for $v \in B_6$; they are shown below.

$$\begin{cases} d(i,8) + d(i+1,8) - d(i,i+1) & = 0 \quad 1 \le i \le 5 \\ d(1,2) + d(1,3) + d(2,3) + d(k,8) - & \\ \sum\limits_{i=1,2,3} (d(i,k) + d(i,8)) & = 0 \quad k = 6,7 \end{cases} \tag{8.3}$$

We already know from (8.2) that $d(i, i+1) = d(1,2) = d(2,3) = 2t$ and $d(i,k) = d(1,3) = t$. From (8.3), we deduce that $d(i,8) = d(k,8) = t$. Hence, $r(B_7, d_0) = 1$ and thus 3_{21} is extreme.

In fact, in a more easy way, we can derive the extremality of the polytope 3_{21} from that of 2_{21}, by using Lemma 6.5 (with $P = 3_{21}, P_1 = 2_{21}$ and choosing e.g. the vector u_{13}^* as v). ■

9. Extreme rays of H_7 from the Gosset polytope 2_{21}

We saw in section 8 that the polytope 2_{21} is a 6-dimensional extreme L-polytope. Therefore, 2_{21} generates extreme rays of the hypermetric cone H_7; namely, for every basic subset B of vertices of 2_{21}, $|B| = 7$ and the hypermetric space (B, d_0) is an extreme ray of H_7. In this section, we wish to investigate how many distinct extreme rays of H_7 arise in this way; by "distinct", we mean "distinct up to permutation" since any permutation of $[1, 7]$ clearly preserves extreme rays. Actually, we believe that all extreme rays of H_7 come from the polytope 2_{21}.

We keep the notation from section 8. The set of vertices of 2_{21} is a two-distance set, the two possible distances being t and $2t$ (setting $t = 2$). One can represent it by a graph G_S whose nodes are the 27 vertices of 2_{21} and there are edges between pairs of vertices at the smallest distance t. This graph is the classical Schläfli graph. For any basic subset B of vertices, let $G(B)$ denote the induced subgraph on B of the Schläfli graph; $G(B)$ is called a *basic* graph. Clearly, if two basic sets have isomorphic basic graphs, then they induce the same (up to permutation) extreme rays. Hence, we are interested in finding all non isomorphic basic subgraphs of the Schläfli graph.

In section 8, we have exhibited a basic subset of 2_{21}; the complement of its basic graph is a cycle of length 5 together with two isolated nodes, shown in Figure 2.

We found seven more basic graphs in connection with the following result by Assouad and Delorme. Assouad and Delorme ([1]) proved that, given a graph G, its suspension ∇G (obtained by adding a new node adjacent to all nodes of G) is hypermetric, but not ℓ_1-embeddable (i.e. the graphic distance induced by ∇G satisfies all hypermetric inequalities but does not belong to the cut cone) if and only if G is an induced subgraph of the Schläfli graph and G contains as an induced subgraph one of the eight graphs $G_i, 1 \leq i \leq 8$, whose complements $\bar{G}_i$ are shown in Figure 4. The graphs G_i are on seven nodes and their graphic distances coincide (up to multiple) with the hypermetric spaces $(V(G_i), d_0)$ (where $V(G_i)$ denotes a subset of vertices of 2_{21} corresponding to the node set of G_i). We saw in section 8 that the graph G_1 has for node set the basic set B_6 and thus gives an extreme ray for H_7. In fact, all graphs $G_i, 1 \leq i \leq 8$, have basic sets as node sets and thus give extreme rays.

By direct inspection of the 7-vertices subgraphs of the Schläfli graph, we found 18 additional affine bases of E_6 and that there is no other nonisomorphic basis. Note that the corresponding hypermetrics are not necessarily graphic. We show in Figure 5 the complements of the basic graphs $G_i, 1 \leq i \leq 26$, of these 26 basic sets (so, in Figure 5, an edge means distance $2t$). Actually, the 26 basic graphs are partitioned in five classes indexed by some integer $q, q = 8, 11, 12, 14, 15$. In fact, all basic graphs of the same class are switching equivalent and the invariant q of each switching class is the number of odd tuples, i.e. triples of nodes carrying an odd number of edges.

Let us explain why the switching operation occurs here. The notion of switching considered here is the well-known notion of graph switching due

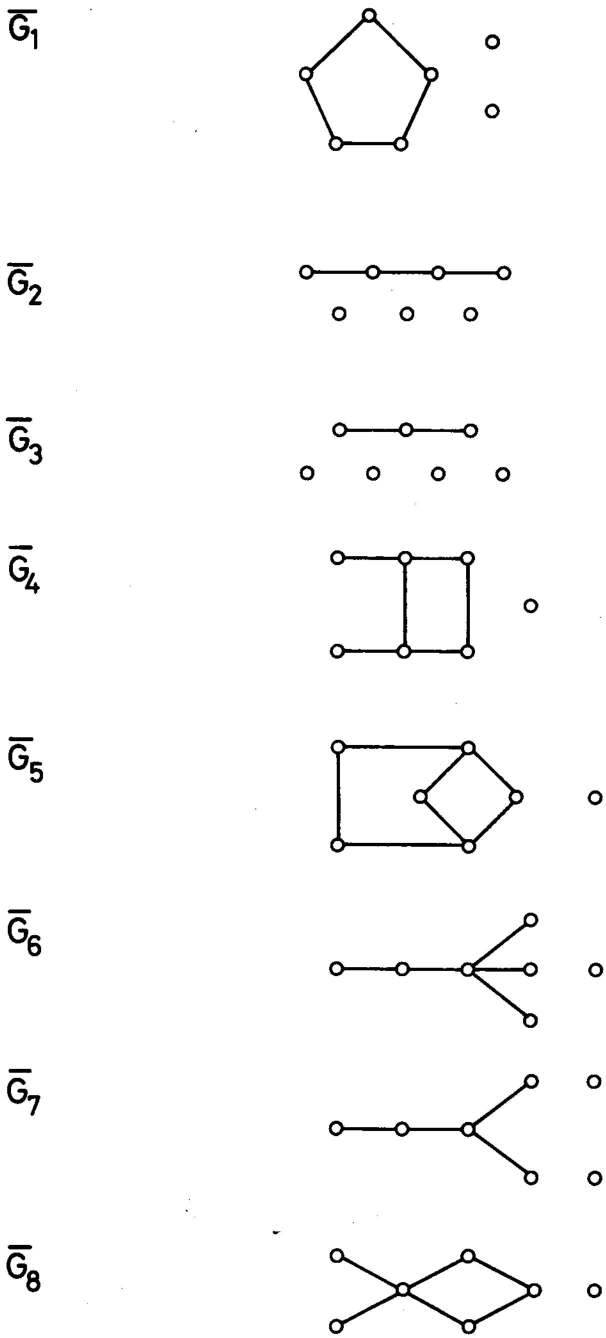
$\overline{G}_1$
$\overline{G}_2$
$\overline{G}_3$
$\overline{G}_4$
$\overline{G}_5$
$\overline{G}_6$
$\overline{G}_7$
$\overline{G}_8$

Figure 4.

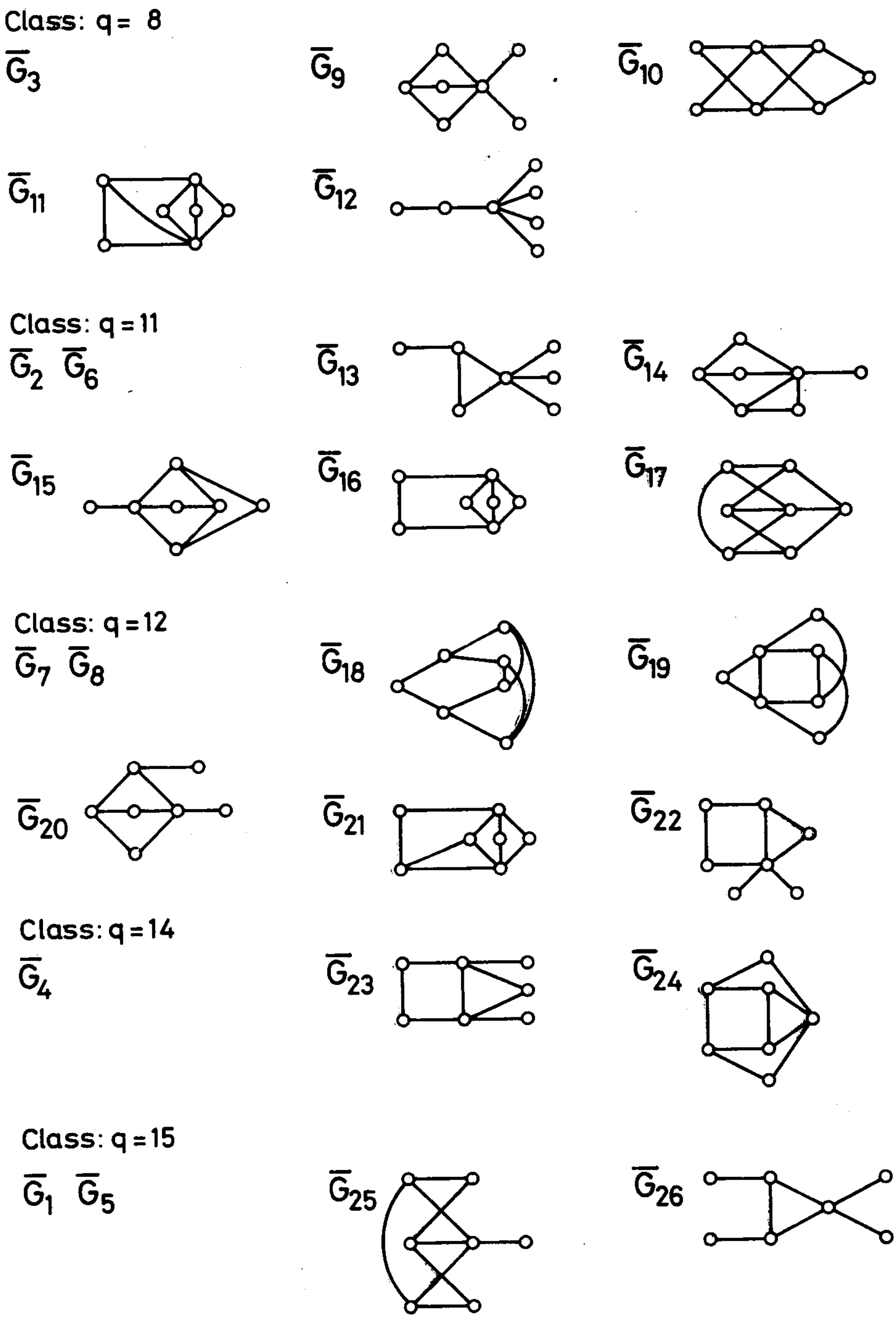
Class: q= 8
$\overline{G}_3$
$\overline{G}_9$
$\overline{G}_{10}$
$\overline{G}_{11}$
$\overline{G}_{12}$
Class: q = 11
$\overline{G}_2$ $\overline{G}_6$
$\overline{G}_{13}$
$\overline{G}_{14}$
$\overline{G}_{15}$
$\overline{G}_{16}$
$\overline{G}_{17}$
Class: q = 12
$\overline{G}_7$ $\overline{G}_8$
$\overline{G}_{18}$
$\overline{G}_{19}$
$\overline{G}_{20}$
$\overline{G}_{21}$
$\overline{G}_{22}$
Class: q = 14
$\overline{G}_4$
$\overline{G}_{23}$
$\overline{G}_{24}$
Class: q = 15
$\overline{G}_1$ $\overline{G}_5$
$\overline{G}_{25}$
$\overline{G}_{26}$

Figure 5.

to Seidel (see e.g. [16]). If $G = (X, E)$ is a graph and $U \subseteq X$ is a subset of nodes of X, the graph obtained by switching G by the set U is the graph $G' = (X, E')$ with $E' = E \Delta \delta(U)$ (the symmetric difference of E and $\delta(U)$), where $\delta(U)$ is the set of edges of G that have exactly one endnode in U. It is well known (see e.g. [16]) that there is a one-to-one correspondence between the switching classes of graphs on n nodes and the sets of n equiangular lines such that $\cos\phi = \frac{1}{\lambda}$, where ϕ is the common acute angle between the lines and $-\lambda$ is the smallest eigenvalue of the ± 1-adjacency matrix of the graphs.

The 56 vertices of the polytope 3_{21} are arranged in 28 pairs of antipodal points (u, u^*). Recall that, for $u, v \in V_7$, the vertex set of 3_{21}, with $u \neq v, v^*$, then,

$$\begin{aligned} d_0(u, v) &= t \text{ (i.e. } u \cdot v = 1) \quad \text{if and only if} \\ d_0(u, v^*) &= 2t \text{ (i.e. } u \cdot v = 0). \end{aligned} \tag{9.1}$$

The 28 pairs of antipodal points compose an equiangular set of 28 lines and, thus, they correspond to the switching class of some graph on 28 nodes. One can construct this graph in the following way. Take a subset V of the set V_7 of vertices of 3_{21} such that V contains no pair of antipodal points and $|V| = 28$ and let $G(V)$ denote the graph on V with edges between points at distance t. From (9.1), it is clear that, if we switch $G(V)$ by the set $U \subseteq V$, we obtain a graph isomorphic to $G(V - U \cup U^*)$, where $U^* = \{u^* : u \in U\}$, i.e. switching by U amounts to replace the points of U by their antipodes.

Since the vertex set V_6 of 2_{21} is contained in the vertex set V_7 of 3_{21}, among the graphs $G(V)$ of the switching class, some of them are the suspension of the Schläfli graph; call such graphs $G(V)$ *feasible*. So, a graph is feasible if it contains the Schläfli graph G_S as an induced subgraph and the remaining vertex, say w, is adjacent to all vertices of G_S; such a graph is denoted by $G_S(w)$. For example, with the notation of section 8, the graph $G(V_6)$ is of the form $G_S(w_0)$.

Switching does not always preserve feasible graphs; if it does, call it a feasible switching. The next Lemma identifies the feasible switchings. Given a vertex v of $G_S(w)$, denote by $N(v)$ the set of vertices that are not adjacent to v in $G_S(w)$.

Lemma 9.1. *Consider a feasible graph $G_S(w)$. Then, its feasible switchings are the switchings by the sets of the form $N(v)$ for some vertex v of $G_S(w)$.*

Proof. It is easy to see that any feasible switching is necessarily a switching by a set $N(v)$ for some vertex v of $G_S(w)$. Let K denote the graph obtained by switching $G_S(w)$ by $N(v)$. Clearly, the vertex v is adjacent to all other vertices in K. In fact, switching $G_S(w)$ by $N(v)$ amounts to replace all vertices $x \in N(v)$ by their antipodes x^*. Thus, the graph K is, in fact, isomorphic to the graph $G_S(w)$, namely, through the mapping of the vertices v, x^* for $x \in N(v), x \notin N(v)$ of K to the vertices $w, x \in N(v), x \notin N(v)$ of $G_S(w)$, respectively. Hence, K is a feasible graph. ■

Let us introduce the following notion of pseudoswitching. Let $G = (X, E)$ be a graph and $N(v)$ denote the set of nodes of X that are not adjacent to v, for $v \in X$. The *pseudoswitching* of G by $N(v)$ is the graph $p_v(G)$ obtained by switching G by $N(v)$ and then deleting all the new edges created between $N(v)$ and v. Therefore, we deduce from the proof of Lemma 9.1 that any pseudoswitching preserves the Schläfli graph.

Consider a basic subgraph $G(B)$ of the feasible graph $G_S(w_0)$, i.e. B is a basic set of the polytope 2_{21}. Given a vertex v of $G_S(w_0)$, apply pseudoswitching to $G_S(w_0)$ by $N(v)$; then, the graph $G(B)$ is transformed in some graph isomorphic to the graph $G(B_v)$ where:

- if $v \notin B$, $B_v = (B - N(v)) \cup (B \cap N(v))^*$
- if $v \in B$, $B_v = (B - N(v)) \cup (B \cap N(v))^* \cup \{w_0\}$

In fact, when $v \notin B$, pseudoswitching simply acts as usual switching on $G(B)$. In both cases, B_v has clearly affine rank 7. The next proposition shows that, in the first case ($v \notin B$), B_v is also an affine basis of E_6, i.e. feasible switching preserves basic sets. However, pseudoswitching does not preserve, in general, basic sets. On the other hand, it can be observed that all 26 basic graphs (from Figure 5) are pseudoswitching equivalent.

In summary, the 26 distinct extreme rays of H_7 coming from 2_{21} arise by (some suitable) pseudoswitchings of a given one and they are further partitioned in five classes of switching equivalent ones.

Proposition 9.2. *Feasible switchings preserve basic sets, i.e. if B is a basic set and $v \notin B$, then $B_v = (B - N(v)) \cup (B \cap N(v))^*$ is also a basic set.*

Proof. Suppose that $G(B)$ is a subgraph of the graph $G_S(w_0)$. In order to show that B_v is an affine basis of E_6, it suffices to verify that the simplices $\Delta(B)$ and $\Delta(B_v)$ generated, respectively, by B and B_v, have the same volume (equal to $\frac{\det(E_6)}{6!}$). Let $\Delta_7(B)$ and $\Delta_8(B)$ denote the simplices generated, respectively, by $B \cup \{\frac{v_0}{2}\}$ ($\frac{v_0}{2}$ is the center of the circumscribed sphere to 3_{21}) and $B \cup \{\frac{v_0}{2}, 0\}$; they are, respectively, of dimension 7 and 8. Then, $\mathrm{vol}(\Delta_8(B)) = \frac{1}{8!}|\det(B \cup \{\frac{v_0}{2}\})|$. Also,

$$\mathrm{vol}(\Delta_8(B)) = \frac{\mathrm{vol}(\Delta_7(B))}{8} \left\| \frac{v_0}{2} \right\|$$

because $\|\frac{v_0}{2}\|$ is the distance of $\frac{v_0}{2}$ to the hyperplane F_7 containing $\Delta_7(B)$. Similarly,

$$\mathrm{vol}(\Delta_7(B)) = \frac{\mathrm{vol}(\Delta(B))}{7} \left\| \frac{v_0}{2} - \frac{v_0 + w_0}{3} \right\|.$$

Therefore,

$$\frac{1}{6!}|\det(B \cup \{\frac{v_0}{2}\})| = \left\| \frac{v_0}{2} \right\| \left\| \frac{v_0}{2} - \frac{v_0 + w_0}{3} \right\| \mathrm{vol}(\Delta(B)).$$

Define, similarly, the simplices $\Delta_7(B_v \cup \{\frac{v_0}{2}\})$ and $\Delta_8(B_v \cup \{\frac{v_0}{2}, 0\})$. Since $G(B_v)$ is now contained in the graph $G_S(v)$, one can compute in the same way the volumes of the simplices and obtain that

$$\frac{1}{6!}|\det(B_v \cup \{\frac{v_0}{2}\})| = \left\| \frac{v_0}{2} \right\| \left\| \frac{v_0}{2} - \frac{v_0 + v}{3} \right\| \mathrm{vol}(\Delta(B_v)).$$

But, $\|\frac{v_0}{2} - \frac{v_0+w_0}{3}\| = \|\frac{v_0}{2} - \frac{v_0+v}{3}\|$, because $v_0 \cdot v = v_0 \cdot w_0 = 1$. One can check that $|\det(B \cup \{\frac{v_0}{2}\})| = |\det(B_v \cup \{\frac{v_0}{2}\})|$ by performing some determinant manipulation, using the fact that $B_v - B$ consists of the vectors $v^* = v_0 - v$ for $v \in B - B_v$. Therefore, $\mathrm{vol}(\Delta(B_v)) = \mathrm{vol}(\Delta(B))$. ■

Finally, note that one obtains at least 26 distinct extreme rays for H_8 from 3_{21}. Indeed, if B is a basic set of 2_{21} and, say, B is contained in $G_S(w_0)$, then $B \cup \{w_0\}$ is a basic set of 3_{21}. We do not know about the classification of all other basic sets.

10. Extreme L-polytopes in the Leech lattice Λ_{24}

In this section, we describe two extreme L-polytopes coming from the Leech lattice Λ_{24}. They have dimension 22, 23 and they are constructed by taking two consecutive suitable sections of the sphere of minimal vectors of Λ_{24}, precisely in the same way as the Gosset polytopes $3_{21}, 2_{21}$ were constructed from the lattice E_8 in section 8.

We refer to [5] for a precise description of the Leech lattice Λ_{24}; we only recall now some facts that we need for our treatment.

The Leech lattice Λ_{24} is a 24-dimensional lattice in $\mathbb{R}^{24}$. For convenience, the coordinates of the vectors $x \in \mathbb{R}^{24}$ are indexed by the elements of $I = \{\infty, 0, 1, \ldots, 22\}$. For $i \in I$, let e_i denote the i-th unit vector whose coordinates are all equal to zero except the i-th one equal to 1. For a subset S of I, set $e_S = \sum_{i \in S} e_i$.

Let $\mathcal{B}_{24}$ denote the family of blocks of the Steiner system $S(5,8,24)$ defined on the set I; hence, $|\mathcal{B}_{24}| = 759$. Set $\mathcal{B}_{23} = \{B - \{\infty\} : B \in \mathcal{B}_{24}$ with $\infty \in B\}$; so $\mathcal{B}_{23}$ is the family of blocks of the Steiner system $S(4,7,23)$ defined on the set $\{0, 1, \ldots, 22\}$ and $|\mathcal{B}_{23}| = 253$. In $\mathcal{B}_{23}$, there are exactly 176 blocks that do not contain a given point and there are exactly 77 blocks that do contain a given point.

The Leech lattice Λ_{24} is generated by the vectors $e_I - 4e_\infty$ and $2e_B$ for all blocks $B \in \mathcal{B}_{24}$. Let V denote the set of minimal vectors of Λ_{24}; so, $x \cdot x = 32$ for $x \in V$. (Note that, in the usual definition, all vectors are scaled by a factor of $\frac{1}{\sqrt{8}}$ and the minimal norm is 4; we choose to omit this factor in order to make the notation easier.) The set V consists of the following vectors:

(I) $(\pm 4^2, 0^{22})$ ($1104 = 2 \cdot 24 \cdot 23$ such vectors)

(II) $(\pm 2^8, 0^{16})$, where the positions of the nonzero components form a block of $\mathcal{B}_{24}$ and there is an even number of minus signs ($2^7 \cdot 759$ such vectors)

(III) $(\mp 3, \pm 1^{23})$, where the ∓ 3 may be in any position, but the lower signs are taken on a codeword of the Golay code $\mathcal{C}_{24}$.

Recall that the codewords of $\mathcal{C}_{24}$ which have exactly 8 nonzero coordinates are precisely the blocks of $\mathcal{B}_{23}$.

Set $c = (5, 1^{23})$ and $a_0 = (4, 4, 0^{22})$; so $c, a_0 \in \Lambda_{24}$, $c \cdot c = 48$ and $a_0 \in V$. Set $V_{23} = \{v \in V : v \cdot c = 24\}$ and $V_{22} = \{v \in V : v \cdot c = 24$ and $v \cdot a_0 = 16\}$. Then, by Lemma 7.2, the polytopes $P_{23} = \mathrm{Conv}(V_{23}), P_{22} = \mathrm{Conv}(V_{22})$ are L-polytopes; they have dimension 23, 22, respectively.

The center of the sphere circumscribing P_{23} is the vector $\frac{c}{2}$. Clearly, $a_0 \in V_{23}$ and its antipode $a_0^* = c - a_0 = (1, -3, 1^{22})$ also belongs to V_{23}; therefore, P_{23} is centrally symmetric. The set V_{23} consists of the vectors a_0, a_0^* together with the following vectors:

(aI) $a_i := (4, 0, 0, \cdots, 4_i, 0, \cdots, 0)$, where the second "4" is in the i-th position, for $1 \leq i \leq 22$, and their antipodes $a_i^* = c - a_i =$

$(1, 1, 1, \cdots, -3_i, 1, \cdots, 1)$ where -3 is in the i-th position, for $1 \leq i \leq 22$.

(aII) $b(S) := (2, 2^7, 0^{16})$, where the first "2" is in the first position (∞) and the positions of the seven other 2's form the block S of $\mathcal{B}_{23}$.

(aIII) $c(T) := (3, -1^7, 1^{16})$, where "3" is in the first position and the positions of the seven -1's form the block T of $\mathcal{B}_{23}$.

Therefore, $|V_{23}| = 2 + 2 \cdot 22 + 2 \cdot 253 = 552$; hence, the polytope P_{23} is centrally symmetric and realizes equality in the bound (7.2).

The set V_{22} consists of the following vectors:

(bI) a_i for $1 \leq i \leq 22$

(bII) $b(S)$ for all blocks S of $\mathcal{B}_{23}$ containing 0

(bIII) $c(T)$ for all blocks T of $\mathcal{B}_{23}$ not containing 0.

Therefore, $|V_{22}| = 22+77+176 = 275$; hence, the polytope P_{22} is asymmetric and realizes equality in the bound (7.1). Note that $V_{23} = V_{22} \cup V_{22}^* \cup \{a_0, a_0^*\}$, where $V_{22}^* = \{v^* : v \in V_{22}\}$.

In fact, both polytopes P_{22}, P_{23} are basic and extreme. We indicate now how to construct an affine basis. We first recall a property of the Steiner system $\mathcal{B}_{23}$.

The set $\{0, 1, \ldots, 22\}$ can be partitioned into two sets A, B such that $0 \in A$, $|A| = 11$, $|B| = 12$ and :

(10.1) For any $i \in A$, there exist two blocks T_i, T_i' of $\mathcal{B}_{23}$ such that $T_i \cap T_i' = \{i\}$ and $T_i \cup T_i' = B \cup \{i\}$.

Namely, we can take: $A = \{0, 1, 3, 4, 5, 8, 10, 11, 12, 17, 21\}$ and $B = \{2, 6, 7, 9, 13, 14, 15, 16, 18, 19, 20, 22\}$ and then:

$$T_0 = \{0, 7, 15, 16, 19, 20, 22\}\ ,\ T_1 = \{1, 6, 7, 9, 13, 15, 22\},$$

$$T_3 = \{2, 3, 9, 14, 15, 16, 22\}\ ,\ T_4 = \{2, 4, 6, 9, 19, 20, 22\},$$

$$T_5 = \{5, 9, 13, 16, 18, 19, 22\}\ ,\ T_8 = \{6, 8, 13, 14, 16, 20, 22\},$$

$$T_{10} = \{7, 9, 10, 14, 18, 20, 22\}\ ,\ T_{11} = \{2, 6, 7, 11, 16, 18, 22\},$$

$$T_{12} = \{2, 12, 13, 15, 18, 20, 22\}\ ,\ T_{17} = \{2, 7, 13, 14, 17, 19, 22\},$$

$$T_{21} = \{6, 14, 15, 18, 19, 21, 22\} \text{ and } T_{21}' = \{2, 7, 9, 13, 16, 20, 21\}.$$

We consider the following set of 23 vectors of V_{22}:

$$\mathcal{B} = \{c(T_i) : i \in A - \{0\}\} \cup \{a_i : i \in B - \{22\}\} \cup \{a_{21}, c(T_{21}')\}.$$

We checked that B is an affine basis for the polytope P_{22}, i.e. generates affinely all 275 vertices of P_{22}. In order to show that the polytope P_{22} is extreme, we have to compute the rank of the system $\mathcal{S}^*(B, d_0)$ (as defined in section 4), which is a system of $252 = 275 - 23$ equations in $\binom{23}{2} = 253$ variables. We computed (using computer) that the rank of the associated matrix is equal to 252. Actually, because of the large size of the problem, a direct computation was impossible; so, we computed the rank modulo p (p prime power) and, in fact, already for p=5, we obtained that the rank is 252. (The rank modulo 2,3 was equal to 230,127, respectively.) Therefore, the polytope P_{22} is extreme.

One can extend B to an affine basis for P_{23}. Namely, the set $B \cup \{b(T_0)^*\}$ is an affine basis for P_{23}, i.e. generates affinely all 552 vectors of V_{23}. Indeed, one can chek that:

$$a_0 = b(T_0') + c(T_1) + c(T_1') + a_1 - b(T_0) - 2b(T_0)^*$$

and thus a_0 is spanned by $B \cup \{b(T_0)^*\}$. Then, $a_0^* = b(T_0) + b(T_0)^* - a_0$ is also spanned by $B \cup \{b(T_0)^*\}$, as well as any v^* for $v \in V_{22}$. Now the extremality of P_{23} follows from that of P_{22}, using Lemma 6.5 (taking P_{23} for P, P_{22} for P_1 and a_0 for v). In conclusion, we have shown:

Theorem 10.1.

(i) *The polytope P_{23} is a centrally symmetric extreme L-polytope of dimension 23 with 552 vertices, hence realizing equality in the bound (7.2)*

(ii) *The polytope P_{22} is an asymmetric extreme L-polytope of dimension 22 with 275 vertices, hence realizing equality in the bound (7.1).*

Observe that the set V_{22} is indeed a spherical two-distance set; namely, the distances between the points of V_{22} take the two values 32 or 48. Also, the 276 lines defined by the 276 pairs of antipodal vertices of the polytope P_{23} are equiangular (with common angle $\arccos(\frac{1}{5})$).

11. Extreme L-polytopes in the Barnes-Wall lattice Λ_{16}

In this section, we describe some more examples of extreme L-polytopes coming from the Barnes-Wall lattice.

We refer to [5] for a precise description of the Barnes-Wall lattice Λ_{16}; we only recall here the necessary facts for our treatment.

The Barnes-Wall lattice Λ_{16} is a 16-dimensional lattice in $\mathbb{R}^{16}$. Let V denote the set of minimal vectors of Λ_{16}. Then, V consists of the following vectors:

(I) 480 vectors of the form: $(\pm 2^2, 0^{14})$, where there are two non zero components equal to 2 or -2

(II) 3840 vectors of the form: $(\pm 1^8, 0^8)$, where the positions of the ± 1's form one of the 30 codewords of weight 8 of the first order Reed-Muller code and there are an even number of minus signs.

We show in Figure 6 a list of 15 codewords of weight 8 of the first order Reed-Muller code; the other 15 codewords of weight 8 are obtained by complementation of the codewords shown in Figure 6.

c_{12}	0 0 1 1 1 1	1 1 1 1	0 0 0 0	0 0
c_{13}	0 1 0 1 1 1	0 0 1 0	1 0 1 0	0 1
c_{14}	0 1 1 0 1 1	0 1 0 0	0 0 1 1	1 0
c_{15}	0 1 1 1 0 1	0 0 0 1	0 1 0 0	1 1
c_{16}	0 1 1 1 1 0	1 0 0 0	1 1 0 1	0 0
c_{23}	1 0 0 1 1 1	0 0 1 0	0 1 0 1	1 0
c_{24}	1 0 1 0 1 1	0 1 0 0	1 1 0 0	0 1
c_{25}	1 0 1 1 0 1	0 0 0 1	1 0 1 1	0 0
c_{26}	1 0 1 1 1 0	1 0 0 0	0 0 1 0	1 1
c_{34}	1 1 0 0 1 1	1 0 0 1	0 1 1 0	0 0
c_{35}	1 1 0 1 0 1	1 1 0 0	0 0 0 1	0 1
c_{36}	1 1 0 1 1 0	0 1 0 1	1 0 0 0	1 0
c_{45}	1 1 1 0 0 1	1 0 1 0	1 0 0 0	1 0
c_{46}	1 1 1 0 1 0	0 0 1 1	0 0 0 1	0 1
c_{56}	1 1 1 1 0 0	0 1 1 0	0 1 1 0	0 0

Figure 6.

Hence, there are 4320 minimal vectors in Λ_{16} and $v \cdot v = 8$ for every minimal vector. (Note that in the usual definition, the minimal norm is 4 and all vectors should be scaled by a factor $\frac{1}{\sqrt{2}}$; we choose to omit this factor in order to make the notation easier.)

Set $a = (2^6, 0^{10})$ (the six "2" are in the first six positions which are precisely the first six positions distinguished in Figure 6). Let S denote the sphere of center $\frac{a}{2}$ and radius $\sqrt{6}$; then, S is an empty sphere in Λ_{16} corresponding to a deep hole (i.e. with maximum radius). The associated L-polytope P, defined by $P = \{v \in \Lambda_{16} : ||v-a||^2 = 6\}$, has exactly 512 vertices that we now describe. Note first that the vectors $0 = (0^{16})$ and $a = (2^6, 0^{10})$ are both vertices of P, since $a \in \Lambda_{16}$ and $||\frac{a}{2}||^2 = 6$; they are, in fact, antipodal on the sphere S. Therefore, P is a centrally symmetric L-polytope. Let $v \in \Lambda_{16}$; v is a vertex of P if and only if $v \cdot a = ||v||^2$ holds. The remaining vertices of P, apart from 0 and a, can be partitioned into the following three classes:

(a) First, those lying in the hyperplane H_a^8 defined by the equation: $x \cdot a = 8$, i.e. those that are minimal vectors; denote their set by V^8. There are 135 such vertices and they are of the form:

(aI) $(2^2, 0^4, 0^{10})$, where the two 2's stay in the first six positions

(aII) $(1^4, 0^2, \pm 1^4, 0^6)$, where the first four 1's stay in the first six positions, i.e. the positions of the ± 1's form one of the 15 codewords shown in Figure 6, and there is an even number of minus signs.

(b) The antipodes of the vectors of V^8; denote their set by V^{16}, so $V^{16} = \{a - v : v \in V^8\}$ and they all lie in the hyperplane H_a^{16} of equation: $x \cdot a = 16$. There are also 135 such vertices and they are of the form:

(bI) $(2^4, 0^2, 0^{10})$, where the two 2's stay in the first six positions

(bII) $(1^4, 2^2, \pm 1^4, 0^6)$, the $1, \pm 1$'s form one of the 15 codewords shown in Figure 6 and there is an even number of minus signs.

(c) The remaining vertices lie in the hyperplane H_a^{12} of equation: $x \cdot a = 12$ and they are of the form $v_1 + v_2$ where v_1 is of type I and v_2 is of type II; denote their set by V^{12}. More precisely, take v_2 of the form $(1^4, 0^2, \pm 1^4, 0^6)$ (there are $15 \times 8 = 120$ such vectors) and v_1 of the form $(2, 0^5, \pm 2, 0^9)$, where the first "2" stays in the two positions of the first two zeros of v_2 and ± 2 stays in one of the positions of the $\pm 1's$ of v_2 and has the opposite sign (there are 8 choices for v_1). For example, for $v_2 = (0,0,1,1,1,1,1,1,1,1,0,0,0,0,0,0)$, $v_1 = (2,0,0,0,0,0,-2,0,0,0,0,0,0,0,0,0)$,we obtain the vector $v = v_1 + v_2 = (2,0,1,1,1,1,-1,1,1,1,0,0,0,0,0,0)$. Note, however, that v can be obtained as the sum of three other pairs of vectors v_2, v_1. Namely,

$$v = (0,0,1,1,1,1,-1,-1,1,1,0^6) + (2,0,0,0,0,0,0,2,0,0,0^6),$$

$$v = (0,0,1,1,1,1,-1,1,-1,1,0^6) + (2,0,0,0,0,0,0,0,2,0,0^6) \text{ and}$$

$$v = (0,0,1,1,1,1,-1,1,1,-1,0^6) + (2,0,0,0,0,0,0,0,0,2,0^6).$$

Therefore, in total, there are $\frac{120\times 8}{4} = 240$ vectors in V^{12} and they are of the form: $(2,0,1^4,\pm 1^4,0^6)$, where the positions of the $1,\pm1$'s form one of the 15 codewords of Figure 6, the "2" stays on one of the two remaining places in the first six positions and there is an odd number of minus signs. These 240 vectors are clearly divided in 120 pairs of antipodal vectors lying respectively in the hyperplanes H_b^2 (of equation: $x \cdot b = 2$) and H_b^{-2} (of equation: $x \cdot b = -2$), where $b = (0^6, 1^{10})$ (H_b^2 contains the vertices with exactly one minus sign and H_b^{-2} contains the vertices with three minus signs).

In summary, the set of vertices of P is: $V = V^8 \cup V^{12} \cup V^{16} \cup \{0, a\}$ and so $|V| = 512$. Therefore, P is a centrally symmetric L-polytope of dimension 16 corresponding to a deep hole of Λ_{16} and having 512 vertices; we will see below that P is basic, hence, P has more vertices than the minimum required for extremality by the bound (7.2). We will see that P is indeed an extreme L-polytope.

In fact, by taking some sections of the empty sphere S by some suitable hyperplanes H_a^α, we can construct some more 15-dimensional L-polytopes, including several examples of extreme ones.

Clearly, the sets $\Lambda_{15}^\alpha = \Lambda_{16} \cap H_a^\alpha = \{x \in \Lambda_{16} : x \cdot a = \alpha\}$, for $\alpha = 8, 12, 16$, are 15-dimensional lattices and they all identical up to translation; note that they are different from the laminated lattice Λ_{15} (see [5]). The sphere $S^\alpha = S \cap H_a^\alpha$ is an empty sphere in the lattice Λ_{15}^α; therefore, the polytope $P^\alpha = \mathrm{Conv}(V^\alpha) = \mathrm{Conv}(S \cap H_a^\alpha)$ is a 15-dimensional L-polytope in Λ_{15}^α, for any $\alpha = 8, 12, 16$.

Both P^8, P^{16} are asymmetric L-polytopes with 135 vertices; hence, they realize equality in the bound (7.1) (as we see below, they are basic). In fact, P^8 is an affine image of P^{16}. The polytope P^{12} is centrally symmetric with 240 vertices; thus, it realizes equality in the bound (7.2). Note however that the set of vertices of P^{16} is not a spherical two-distance set (indeed, there are three possible distances between the vertices of P^{16}: 8,12,16); also, the 120 lines defined by the 120 pairs of antipodal vertices of P^{12} are not equiangular (there are two possible angles: arccos(0),arccos($\frac{1}{3}$)).

Theorem 11.1.

(i) The polytope P (associated with a typical deep hole of the Barnes-Wall lattice Λ_{16}) is a centrally symmetric extreme L-polytope of dimension 16 with 512 vertices.

(ii) *The polytopes P^8, P^{16} are asymmetric extreme L- polytopes of dimension 15, each having 135 vertices.*

(iii) *The polytope P^{12} is not extreme.*

Proof. We first show (ii), i.e. that P^{16} is extreme. For this, we note first that P^{16} is basic. Indeed, we can find 16 vertices of P^{16} forming an affine basis B of the lattice Λ_{15}^{16}, i.e. such that B generates the set of vertices of P^{16}. We choose as basis the set:

$$B = \{v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{23}\}$$

$$\cup\{c_{12}(13), c_{13}(24), c_{14}(24), c_{15}(\emptyset), c_{23}(24), c_{25}(12)\}$$

$$\cup\{c_{26}(14), c_{34}(34), c_{35}(24), c_{45}(23)\}$$

where v_{ij} denotes the vector $(2^4, 0^2, 0^{10})$ of type (bI) with i,j denoting the positions of the first two 0's, and $c_{ij}(xy)$ denotes the vector obtained from the codeword c_{ij} (see Figure 6) by assigning a minus sign to the 1's in the x-th and y-th positions among the last four 1's of c_{ij}. For example, $c_{12}(13) = (0,0,1,1,1,1;-1,1,-1,1;0,0,0,0;0,0)$ and $c_{15}(\emptyset) = c_{15}$ (no minus sign at all).

One can check that all other vertices of P^{16} are integer affine combinations of the vectors of B. From these combinations, we deduce (as indicated in Remark 4.5), the explicit description of the system of hypermetric equations $\mathcal{S}(B, d_0)$ (consisting of 135-16=119 equations in $\binom{16}{2} = 120$ variables). We checked (using computer) that the set of solutions to this system has rank one, implying that P^{16} is extreme.

We now show (i). Consider the vector

$$v_0 = (2,0,1,1,1,1;-1,1,1,1;0,0,0,0;0,0);$$

v_0 is a vertex of P lying in H_a^{12} (having the shape of the codeword c_{12}). One can check that the set $B \cup \{v_0\}$ forms an affine basis of Λ_{16}, i.e. generates the vertices of P. Using Lemma 6.5 (taking P for P, P^{16} for P_1 and the vector $v_0^* = a - v_0$ for v), we deduce that P is extreme, since P^{16} is extreme.

We prove (iii). Consider the subset X of the vertices of P^{12} that lie in the hyperplane H_b^2; there are exactly 120 such vertices. The polytope Conv(X) is a 14-dimensional asymmetric L-polytope in the lattice $\Lambda_{16} \cap H_a^{12} \cap H_b^2$; but, we checked that it is not extreme, in fact, its rank is equal to 35. Therefore, the L-polytope P^{12} is not extreme (this can be easily seen, using the argument of the proof of Theorem 6.2). ■

Note that the hole of the lattice Λ_{15}^{16} corresponding to the extreme L-polytope P^{16} is not a deep hole; indeed, its radius is equal to $\frac{4}{\sqrt{3}}$, while the radius of the hole of Λ_{15}^{12} corresponding to the L-polytope P^{12} is equal to $\sqrt{6}$ and $6 > \frac{16}{3}$.

We can construct another extreme L-polytope in $\mathbb{R}^{16}$ as follows. Consider the polytope Q whose vertices are the vertices of P that satisfy $x \cdot a = 0, 8, 16$ or 24, i.e. they are the vertices of P^8, or of P^{16}, or they are 0 or a. Hence, Q has $2 \times 135 + 2 = 272$ vertices, Q is a 16-dimensional polytope and the set $B \cup \{a\}$ generates all vertices of Q (B is the set defined in the proof of Theorem 10.1). In fact, Q is an L-polytope in the lattice $\Lambda'_{16} = \Lambda_{16} \cap \{x : x \cdot a = 0(mod 8)\}$; so, Λ'_{16} is the sublattice of Λ_{16} having points only in the layers $x \cdot a = 0, 8, 16, 24, etc.$ Hence, Q is centrally symmetric and realizes equality in the bound (7.2).

Theorem 11.2. *The polytope Q is a centrally symmetric extreme L-polytope of dimension 16 with 272 vertices.*

Proof. We use again Lemma 6.5, taking the polytope Q for P, the polytope P^{16} for P_1 and the vector $0 = a^*$ for v; since P^{16} is extreme, then Q too is extreme. ■

Finally, let us look at some L-polytope obtained by taking a section of the minimal vectors by some hyperplane (as indicated in Lemma 7.2 and similarly to the construction of the Gosset polytopes in section 8 or of the polytopes in the Leech lattice in section 10). Namely, we consider the section by the hyperplane H_a^4 of equation $x \cdot a = 4$. In this way, we obtain the L-polytope $Q' = \text{Conv}(x \in \Lambda_{16} : x \cdot x = 8 \text{ and } x \cdot a = 4)$. Q' is a 15-dimensional L-polytope and it has many vertices; it has 1080 vertices that are of the form:

(i) $(2, 0^5, \pm 2, 0^{10})$, where the first "2" stays in the first six positions (120 such vectors)

(ii) $(\pm 1^4, 0^2, \pm 1^4, 0^6)$, where the positions of the ± 1's form one of the 15 codewords of Figure 6, there is exactly one minus sign in the first four ± 1 and there is an odd number of minus signs in the last four ± 1 (480 such vectors)

(iii) $(1^2, 0^4, \pm 1^6, 0^4)$, where the positions of the 0's form one of the 15 codewords of Figure 6 and there is an even number of minus signs (480 such vectors).

Consider a special vertex $c = (2, 0, .., 0, 2)$ of Q'. One checks easily that the distances $||v - c||^2$ from the other vertices to c take the values $8, 12, 16, 20, 24$; in fact, value 8 (respectively, 12,16,20,24) is taken for 119 (respectively, 336, 427,176,21) vertices of Q'. Therefore, the set of the 119 vertices that are at distance 8 from c forms a 14-dimensional asymmetric L-polytope which realizes equality in the bound (7.1). However, we checked that this polytope is not extreme. On the other hand, we checked that the polytope Q' is extreme.

We summarize in Figure 7 the results from section 11 about the L-polytopes constructed from the Barnes-Wall lattice Λ_{16}. Recall that $a = (2^6, 0^{10})$, $c = (2, 0^{14}, 2)$, S denotes the deep hole of Λ_{16} with center $\frac{a}{2}$ and H_a^α denotes the hyperplane: $x.a = \alpha$.

L-polytope	dimen.	num. of vertices	asymmetric (A) or centrally symmetric (CS)	equality in bound (7.1) or (7.2) ?	extreme ?
$P = \text{conv}(S \cap \Lambda_{16})$	16	512	CS	No	Yes
$P^8 = \text{conv}(S \cap \Lambda_{16} \cap H_a^8)$	15	135	A	Yes	Yes
$P^{16} = \text{conv}(S \cap \Lambda_{16} \cap H_a^{16})$	15	135	A	Yes	Yes
$P^{12} = \text{conv}(S \cap \Lambda_{16} \cap H_a^{12})$	15	240	CS	Yes	No
$Q = \text{conv}(S \cap \Lambda_{16} \cap \{x : x.a = 0, 8, 16, 24\})$	16	272	CS	Yes	Yes
$\text{conv}(x \in \Lambda_{16} : x.x = 8, a.x = 4, x.c = 8)$	14	119	A	Yes	No
$Q' = \text{conv}(x \in \Lambda_{16} : x.x = 8, a.x = 4)$	15	1080	A	No	Yes

Figure 7.

References

[1] P. Assouad and C. Delorme, Graphes plongeables dans L^1, *Comptes Rendus de l'Académie des Sciences de Paris* **291**(1980), 369–372.

[2] P. Assouad, Sous espaces de L^1 et inégalités hypermétriques, *Comptes Rendus de l'Académie des Sciences de Paris* **294**(1982), 439–442.

[3] D. Avis and Mutt, All the facets of the six points Hamming cone, *European Journal of Combinatorics* **10**(1989), 309–312.

[4] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance regular graphs*, Springer Verlag, 1989.

[5] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups,* Grundlehren der Mathematischen Wissenschaften 290, Springer Verlag 1987.

[6] H. S. M. Coxeter, The polytope 2_{21} whose twenty seven vertices correspond to the lines on the general cubic surface, *American Journal of Mathematics* **62**(1940) 457–485.

[7] H. S. M. Coxeter, *Regular polytopes,* Dover Publications Inc. New York.

[8] P. Delsarte, J. M. Goethals and J. J. Seidel, Spherical codes and designs, *Geometriae Dedicata* **6**(1977), 363–388.

[9] M. Deza (Tylkin), On the Hamming geometry of unitary cubes, Doklady Akad. Nauk. SSSR (in Russian) vol. 134 (1960), 1037–1040; English translation in *Soviet Physics Dokl.* **5**(1961), 940–943.

[10] M. Deza, V. P. Grishukhin and M. Laurent, The hypermetric cone is polyhedral, to appear in *Combinatorica.*

[11] M. Deza and M. Laurent, Facets for the cut cone I, Research memorandum RMI 88-13, University of Tokyo (Japan) (1988), to appear in *Mathematical Programming.*

[12] R.M. Erdahl, A cone of inhomogeneous second order polynomials, Report of Queens University, Canada (1990).

[13] P. W. H. Lemmens and J. J. Seidel, Equiangular lines,*Journal of Algebra* **24**(1973), 494–512.

[14] A. Neumaier, Distance matrices, dimension, and conference graphs, *Proceedings of the Koninklijke Nederlande der Academie van Wettenschappen* **A–84**(4)(1981), 385–391.

[15] A. Neumaier, Some relations between roots, holes and pillars, *European Journal of Combinatorics* **8**(1987), 29–33.

[16] J. J. Seidel, Discrete non-euclidian geometry, Handbook of Incidence Geometry, Chapter 21, to appear.

[17] P. Terwilliger and M. Deza, Classification of finite connected hypermetric spaces, *Graphs and Combinatorics* **3**(1987), 293–298.

[18] G. F. Voronoi, Nouvelles applications des paramètres continus à la théorie des formes quadratiques, Deuxième mémoire, *J. Reine Angew. Math.* **134** (1908), 198–287, **136**(1909), 67–178.

Michel Deza
CNRS,
17 Passage de l'Industrie,
75010 Paris, France

Viatcheslav P. Grishukhin
CEMI,
Academy of Sciences of USSR,
Krasikova 32, Moscow 117 418,
USSR

Monique Laurent
CNRS-LAMSADE,
Université Paris Dauphine,
Pl. du Maréchal de Lattre de Tassigny,
75 775 Paris Cedex 16,
France