

Partition Theorems for the Power Set

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ABSTRACT

The power set of any set can be colored by countably many colors such that there do not exist infinitely many disjoint subsets with all finite subunions getting the same color. Two colors suffice for the case of infinite subunions or when uncountably many subsets are required. If GCH holds, $\text{cf}(\lambda) > \kappa$, and the power set of a set of size λ is κ -colored, then there exist λ disjoint monochromatic sets such that their union also gets the same color.

0. Introduction

The long-standing conjecture of R. Graham and B. Rothschild was finally resolved by N. Hindman who in [10] proved that if the collection of all finite subsets of the set of natural numbers is partitioned into finitely many pieces then some piece contains infinitely many disjoint sets with all their finite subunions. The rather involved original proof was later simplified by J. E. Baumgartner [1] and by Glazer (see e. g. in [8], which gives an excellent overview of the whole theory). In [6] P. Erdős asked if higher cardinal versions of Hindman's theorem are true. Specifically, he asked if a large enough set has the property that whenever its subsets are colored by ω colors, then there are disjoint subsets A_i , $i < \omega$, such that all finite unions get the same color. Or, with only two colors, can we find sets as above

with all infinite subunions getting the same color? Already in [6], Erdős expressed the opinion that the answer is probably negative. Some results (see [7,11,14]) partially confirmed this hope, at least, if the ground set is not too large. As for the second statement, the consistency of a negative answer was given by W. Weiss [12,13] (under the axiom of constructibility, see also [9]) and by Bregman, Shapirovski, and Shostak (if $2^{\aleph_0} = \aleph_2$ but otherwise GCH is true), see [2]. Here, with a different method, we show that the answer is, in fact, negative, for both questions.

In the second part of the paper we investigate the question when only the union of all sets is required to have the same color with all the sets. Without the disjointness proviso, several results were proved in [3,4,5]. Recently, H. Lefmann asked if $P(\omega_1)$ is colored with finitely many colors, are there necessarily countably many disjoint monochromatic sets such that their union also shares the same color. This was answered in the affirmative by P. Erdős. Here we generalize this result, i. e. we show that if λ, κ are cardinals, $\text{cf}(\lambda) > \kappa$, and $P(\lambda)$ is colored by κ colors, then there exist disjoint nonempty subsets $\{A_\alpha : \alpha < \lambda\}$ such that all the A_α -s as well as $\cup\{A_\alpha : \alpha < \lambda\}$ get the same color (but some form of GCH is assumed).

1. The negative relations

Theorem 1. *If S is a set, then $P(S)$ can be decomposed into ω classes such that no class contains all finite unions of infinitely many disjoint sets.*

Proof. We identify every $X \subseteq S$ with its well-known characteristic function $\chi_X : S \rightarrow 2$. These functions form a vector space over the two-element field. Let $B = \{b_i : i \in I\}$ be a basis of this space. Put $X \subseteq S$ into class n iff χ_X is the sum of n elements of B . We show that this decomposition works. Assume that A_0, A_1, \dots are disjoint subsets of S . Observe that in our vector space, addition means taking symmetric difference, which is the same as taking unions, when the sets are disjoint. So assume, that all finite sums of the vectors χ_{A_r} are in class n , for $r = 0, 1, \dots$. Put $\chi_{A_r} = \sum\{b_i : i \in E_r\}$ with $E_r \subseteq B$. Clearly, $|E_r| = n$. By the Δ -system lemma, we can assume that $E_r \cap E_s = E$ for $r \neq s$. As $\chi_{A_r} + \chi_{A_s}$ is in class n , $|E| = n/2$. But then, for r, s, t different, the sum

$$\chi_{A_r} + \chi_{A_s} + \chi_{A_t} = \sum\{b_i : i \in E_r \cup E_s \cup E_t\}$$

is in class $2n$, a contradiction. ■

Theorem 2. *If S is a set, then $P(S)$ can be decomposed into two classes such that no class contains all finite unions of some uncountably many pairwise disjoint subsets.*

Proof. We use the same idea as in the proof in Theorem 1. If X is a subset of S , and the vector assigned to X is the union of n elements in the basis, then put X into classes according to the parity of x , where $n = 2^x(2y + 1)$. Assume that $\{A_\alpha : \alpha < \omega_1\}$ are disjoint subsets of S , and all finite unions are in the same class. As above, let $\chi_{A_\alpha} = \sum\{b_i : i \in E_\alpha\}$ for some finite sets E_α . We can assume that these sets form a Δ -system, with kernel E , where $|E| = r$. Then, the union of two E_α -s is represented by the sum of $2(n - r)$ basis elements, while the union of four sets is represented by $4(n - r)$ basis elements, so they are in different classes, as claimed. ■

Theorem 3. *If S is a set, then $P(S)$ is the union of two classes, none containing all infinite unions of some infinitely many subsets of S .*

Proof. The decomposition is the same as in Theorem 2. Assume that $\{A_i : i < \omega\}$ are subsets, such that for every infinite $X \subseteq \omega$, the set $A(X) = \cup\{A_i : i \in X\}$ is in the same class. For every such X , the subset $A(X)$ is represented by the basis elements in, say, $E(X)$, as in the above proof. There is an uncountable family \mathcal{H} of infinite subsets of ω , such that if $A, B \in \mathcal{H}$, then either $A \subseteq B$, or $B \subseteq A$. We can assume that the corresponding $E(X)$ -s form a Δ -system of n -element sets, with an r -element kernel. Take four of them, $X_0 \subset X_1 \subset X_2 \subset X_3$. Then, $A(X_1 - X_0)$ and $A(X_3 - X_2)$ are in the class corresponding to the exponent of 2 in $2(n - r)$, while the union of them is in the class corresponding to the exponent of 2 in $4(n - r)$, so these sets are in different classes. ■

2. The positive relations

Theorem 4. *If $\kappa \geq \omega$ is a cardinal, $2 \leq k < \omega$, and $P(2^\kappa)$ is decomposed into κ classes, then some class contains the nonempty disjoint sets A_1, \dots, A_k as well as their union $\cup\{A_i : 1 \leq i \leq k\}$.*

Proof. We start with a well-known statement.

Lemma. There is an ordered set $(X, <)$ of size $> 2^\kappa$ with a dense subset Y of size 2^κ .

Proof. Let λ be the least cardinal with $2^\lambda > 2^\kappa$. Clearly, $\lambda \leq 2^\kappa$. Let X be the lexicographically ordered set of all $\lambda \rightarrow 2$ functions, let $Y \subseteq X$ be the set of eventually zero functions. ■

Returning to the proof of the Theorem, if $x < y$ are in X , put $A(x, y) = \{z \in Y : x < z \leq y\}$. The coloring of $P(Y)$ colors all sets of type $A(x, y)$, i. e. all pairs of X , so by the Erdős-Rado theorem, there is a homogeneous set of size $k + 1$. If $x_0 < \dots < x_k$ are the points of this set, then $A(x_0, x_1), \dots, A(x_{k-1}, x_k)$ are k disjoint sets in the same class, and their union, $A(x_0, x_k)$ is in the same class, too. ■

In this result 2^κ can not be replaced by κ^+ .

Theorem 5. If $2^\kappa = 2^\omega$, then $P(\kappa)$ can be partitioned into ω classes such that no class contains two disjoint sets together with their union.

Proof. It suffices to show that if V is a vector space over $GF(2)$ of size $\leq 2^\omega$ then there is a coloring $V \rightarrow \omega$ with no monochromatic solution of $x + y = z$. Let $B \subseteq V$ be a basis, and assume that $B = \{b_i : i \in I\}$, where, without loss of generality, $I \subseteq \mathbb{R}$. If $v \in V$, $v = b_{i_1} + \dots + b_{i_n}$, with $i_1 < \dots < i_n$, then color v with the ordered sequence (q_1, \dots, q_{n-1}) , where q_1, \dots, q_{n-1} are rational numbers with $i_1 < q_1 < \dots < q_{n-1} < i_n$. Assume that x, y, z get the same color, and $x + y = z$. Then, for some n , the vectors x, y , and z are all the sum of n elements of the basis, and the same sequence of rational numbers separates the indices of those basis elements. Some b_i must occur both in x and y , but then, by the separability condition, they get the same index, say i . This implies that b_i does not appear in $z = x + y$. However, the i -th component of $x + y$ must occur either in x or y and must be different from the above b_i , a contradiction. ■

Theorem 6. (GCH) Assume that $\text{cf}(\lambda) > \kappa \geq \omega$. Then, if $P(\lambda)$ is colored with κ colors, then there exist disjoint, monocolored sets X_α , ($\alpha < \lambda$), such that their union gets the same color, as well.

Proof. If λ is regular, put $A = \lambda$, and let I be the ideal of the non-stationary sets on λ . If λ is singular, let $\{\lambda_\alpha : \alpha < \text{cf}(\lambda)\}$ be a strictly increasing sequence of cardinals converging to λ with $\lambda_0 > \kappa$. Put $\kappa_\alpha = \lambda_\alpha^+$, let $\{A_\alpha : \alpha < \text{cf}(\lambda)\}$ be disjoint sets with $|A_\alpha| = \kappa_\alpha$, and put $A = \cup\{A_\alpha : \alpha < \text{cf}(\lambda)\}$.

$\alpha < \text{cf}(\lambda)\}$ as the ground set, rather than λ . For $B \subseteq A$, put $B \in I$ iff $|B \cap A_\alpha| \leq \lambda_\alpha$ holds for all $\alpha < \text{cf}(\lambda)$. Clearly, in both cases, I is a κ^+ -complete ideal, on A . We need two more properties of I .

Claim 1. *Less than λ members of I can not cover A .*

Proof. Straightforward, if λ is regular. If λ is singular, then λ_α members of I can not cover A_α , let alone A . ■

Claim 2. *Assume that $X_\xi \in I$ are disjoint sets ($\xi < \lambda$). Then there is a decomposition $\lambda = \cup\{Y_\alpha : \alpha < \lambda\}$ into disjoint λ -sized sets, such that $\cup\{X_\xi : \xi \in Y_\alpha\} \in I$ for every $\alpha < \lambda$.*

Proof. It suffices to find disjoint λ -sized sets $\{Y_\alpha : \alpha < \lambda\} \subseteq P(A)$ with $\cup\{X_\xi : \xi \in Y_\alpha\} \in I$, as the remaining elements of $\lambda - \cup\{Y_\alpha : \alpha < \lambda\}$ can be added to distinct groups, so the unions will still be in I . As $\lambda^2 = \lambda$, this latter statement can further be reduced to showing that the union of some λ members of the family is in I .

Assume first, that λ is regular, so the sets are non-stationary. We may assume, that they are ordered by their first elements, i. e. if $\gamma(\xi) = \min(X_\xi)$, then $\xi_0 < \xi_1$ implies $\gamma(\xi_0) < \gamma(\xi_1)$, and so we have $\gamma(\xi) \geq \xi$ for $\xi < \lambda$. Put $Y = \{\alpha + 1 : \alpha < \lambda\}$. If $y \in Z = \cup\{X_\xi : \xi \in Y\}$, put $f(y) = \xi$ for $y \in X_\xi$. Assume that Z is stationary (as otherwise we are done). $f(y) \leq y$, as $\gamma(\xi) \geq \xi$. The range of f consists of successor ordinals, a non-stationary set. On $Z - Y$, a stationary set, $f(y) < y$ holds, so by Fodor's theorem, $f(y) = \xi$ on a stationary set, for some ξ , a contradiction, as the X_ξ 's are non-stationary.

Assume now that λ is singular. As our sets X_ξ are disjoint, the set

$$B_\alpha = \{\xi < \lambda : X_\xi \cap \cup\{A_\beta : \beta < \alpha\} \neq \emptyset\}$$

is of size $\leq \lambda_\alpha$ for $\alpha < \text{cf}(\lambda)$. By transfinite recursion, select $Y_\alpha \subseteq \lambda - B_\alpha$ of size κ_α . Clearly, $Y = \cup\{Y_\alpha : \alpha < \text{cf}(\lambda)\}$ is of size λ . Also, if $Z = \cup\{X_\xi : \xi \in Y\}$, then

$$A_\alpha \cap Z = A_\alpha \cap \cup\{X_\xi : \xi \in \cup\{Z_\beta : \beta < \alpha\}\}$$

is of size $\leq \lambda_\alpha^2 = \lambda_\alpha$, so $Z \in I$. ■

To prove the Theorem, assume that a coloring $P(A) \rightarrow \kappa$ is given, with no monochromatic configuration, as in the statement of the Theorem.

We are going to build a tree, or rather, a ramification system, T . Every element of T will be of the form (B, W) , where B is a subset of A , and W is an equivalence relation on B , such that every equivalence class is in I . If $(B, W) \leq (B', W')$ holds in the tree, then B' will be the union of some classes of W while every class of W' will be the union of some classes of W .

To start, let (A, id_A) be the only element of T_0 , the lowest level of T . If an element $(B, W) \in T_\alpha$ is given, we extend it only if B is not the union of $< \lambda$ members of I . In this case, let $\{X_j : j \in J\}$ be a maximal system of disjoint, non-empty, α -colored sets, such that each X_j is the union of some classes of W . The two immediate extensions of (B, W) will be of the form (B', W') , (B'', W'') , where $B' = \cup\{X_j : j \in J\}$, $B'' = B - B'$. If $|J| = \lambda$, decompose, by Claim 2, J into λ sets of size λ , $J = \cup\{J_\tau : \tau < \lambda\}$, such that $\widehat{J}_\tau = \cup\{X_j : j \in J_\tau\} \in I$ for $\tau < \lambda$. Let $\{\widehat{J}_\tau : \tau < \lambda\}$ be the classes of W' . Put $W'' = W|B''$. Notice, that, if X is either the union of some classes of W' or is the union of some classes of W'' , then the color of X is not α . If $\alpha < \kappa$ is a limit ordinal and (B_β, W_β) form an α -branch in the tree, then we extend this branch as follows. Put $B = \cap\{B_\beta : \beta < \alpha\}$, it will be the intersection of some sets, each being the union of some classes of W_β , so B itself is the union of some classes by W_β , for any $\beta < \alpha$. For $x, y \in B$, put xWy , if $xW_\beta y$ holds for large enough $\beta < \alpha$. Clearly, W is an equivalence relation, and an easy argument shows that $(B_\beta, W_\beta) \leq (B, W)$ holds for every $\beta < \alpha$.

Eventually we get a tree with at most $2^{<\kappa} = \kappa$ nodes (this is the point where GCH is used). By Claim 1, there is a point $x \in A$ which is uncovered by those nodes $(B, W) \in T$ where B is the union of less than λ sets in I . Consequently, for every $\alpha < \kappa$, there is a unique $(B_\alpha, W_\alpha) \in T_\alpha$ with $x \in B_\alpha$. The sequence of sets $\{B_\alpha : \alpha < \kappa\}$ is descending and none of them is the union of less than λ elements in I . The non-empty set $B = \cap\{B_\alpha : \alpha < \kappa\}$ is the union of some classes by W_α , and by our above considerations, B cannot get color α , for any $\alpha < \kappa$, a contradiction. ■

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