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On the Intersection of Independence Systems

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ABSTRACT

A family of independence systems on the same finite underlying set is said to have the *minimax property* if the maximum weight of a common independent set is characterized by a minimax formula which is a direct generalization of the formula of the Matroid Intersection Theorem. A necessary and sufficient condition is given for any family of independence systems to have the minimax property. It is also shown: the minimax property implies that the maximum weight of a common independent set can be found by a polinomial-time algorithm.

1. Introduction

It is an important problem in combinatorial optimization to characterize the maximum size (or weight) of a common independent set of two or more independence system defined on the same finite underlying set. A celebrated example when it is well solved is the (Two-) Matroid Intersection Theorem of Edmonds [1]. On the other hand, the problem is NP-hard in the general case even for three matroids (see e.g. [3]).

It is an exciting question, what makes the problem easy or hard, respectively, depending on the structure and/or the number of independence systems. Although we do not know any complete answer, it is possible to give a nontrivial necessary and sufficient condition characterizing those families of independence systems in which the maximum weight of a common independent set satisfies an Edmonds-type minimax formula. As we shall see below, this fact has consequences for the algorithmic complexity, as well.

2. Definitions, problem statement

Let S be a finite nonempty set of cardinality n. A family $\mathcal{F}$ of subsets of S is called an *independence system* if $\emptyset \in \mathcal{F}$ and for any $A, B \subseteq S$, $A \subseteq B \in \mathcal{F}$ implies $A \in \mathcal{F}$. The elements of $\mathcal{F}$ are called *independent sets.* The *rank function* $r : 2^S \to \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ is defined by

$$r(A) = \max\{|X| : X \subseteq A,\ X \in \mathcal{F}\} \quad (A \subseteq S),$$

that is, $r(A)$ is the size of a maximum independent set contained in A.

An independence system is a *matroid* if its rank function is *submodular,* that is,

$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$$

holds for any $A, B \subseteq S$. There are many other equivalent definitions for matroids (for a general reference see e.g. [7])

If $\mathcal{F}_1, \mathcal{F}_2$ are two matroids on S with rank function r_1, r_2, respectively, then the Matroid Intersection Theorem [1] says that

$$\max\{|A| : A \in \mathcal{F}_1 \cup \mathcal{F}_2\} = \min\{r_1(X) + r_2(S - X) : X \subseteq S\}$$

holds.

In what follows we consider the more general weighted case with non-negative integer weights on the elements of S. For that purpose the *weighted rank function* $r : \mathbb{Z}_+^n \to \mathbb{Z}_+$ is defined by

$$r(x) = \max\{x(A) : A \in \mathcal{F}\} \quad (x \in \mathbb{Z}_+^n),$$

where $x(A)$ denotes the weight of the set A, which is the sum of the weights of its elements with respect to the weighting vector $x \in \mathbb{Z}_+^n$. Clearly, if x is restricted to 0–1 vectors, then we get back the unweighted rank function. As no ambiguity arises, the same letter is used for both (even if the weights are allowed to be arbitrary reals).

The weighted case of the Matroid Intersection Theorem says that for any weighting vector $w \in \mathbb{Z}_+^n$

$$\max\{w(A) : A \in \mathcal{F}_1 \cup \mathcal{F}_2\} = \min\{r_1(x) + r_2(y) : x, y \in \mathbb{Z}_+^n, \quad x + y = w\} \tag{1}$$

holds.

Let $\mathcal{F}_1, \dots, \mathcal{F}_m$ be general independence systems on S. It is not difficult to see that a direct but weak generalization of (1) can always be stated for any weighting $w \in \mathbb{Z}_+^n$:

$$\max\{w(A) : A \in \mathcal{F}_1 \cap \ldots \cap \mathcal{F}_m\} \leq \tag{2}$$

$$\leq \min\{r_1(x_1) + \ldots + r_m(x_m) : x_1, \ldots, x_m \in \mathbb{Z}_+^n, \quad x_1 + \ldots + x_m = w\}.$$

We are interested in the question: what conditions can guarantee *equality* in (2)? To give a name to this property let us introduce a definition:

Definition. *The family* $(\mathcal{F}_1, \dots, \mathcal{F}_m)$ *of independence systems has the* minimax property *if equality holds in (2) for any weighting* $w \in \mathbb{Z}_+^n$.

Let us call a weighting vector $w \in \mathbb{Z}_+^n$ *even* if each component of w is an even number. Now it is clear that the left-hand side of (2) is an even number for any even weighting $w \in \mathbb{Z}_+^n$ (as then every set has an even weight). So in order to have equality in (2), the right-hand side must be also an even number in this case. Thus, an obvious *necessary* condition for having the minimax property is that the minimum on the right-hand side of (2) should be an even number for any even weighting $w \in \mathbb{Z}_+^n$. The main result is that it is also *sufficient,* which gives a characterization of the families of independence systems having the minimax property.

3. Results

Theorem 1. *The family* $(\mathcal{F}_1, \dots, \mathcal{F}_m)$ *of independence systems has the minimax property if and only if the (integer) value*

$$f(w) = \min\{r_1(x_1) + \ldots + r_m(x_m) : x_1, \ldots, x_m \in \mathbb{Z}_+^n, \\ x_1 + \ldots + x_m = w\} \quad (3)$$

is an even number for any even weighting $w \in \mathbb{Z}_+^n$.

Proof. Let P_i be the polytope defined as the convex hull of independent sets of $\mathcal{F}_i$ $(i = 1, 2, \ldots, m)$ (the sets are identified with their characteristic vectors). Then clearly

$$r_i(y) = \max\{yx : x \in P_i\}$$

holds for any $y \in \mathbb{R}^n$ (extending the rank function to arbitrary real weights in the obvious way)

Define the function $g : \mathbb{Z}_+^n \to \mathbb{R}$ by

$$g(w) = \inf\{r_1(x_1) + \ldots + r_m(x_m) : x_1, \ldots, x_m \in \mathbb{R}^n, \\ x_1 + \ldots + x_m = w\}, \quad (4)$$

that is, the condition $x_1, \ldots x_m \in \mathbb{Z}_+^n$ in (3) is relaxed to $\mathbb{R}^n$. First we show that $g(w) \equiv f(w)$.

If some x_i in (4) has a negative component, then changing this component to zero will not increase $r_i(x_i)$ and decreasing some components in the other x_j's (to restore $x_1 + \ldots + x_m = w$) also will not increase the value of $r_1(x_1) + \ldots + r_m(x_m)$. So it is enough to restrict ourselves to nonnegative x_i's, that is, one can replace $\mathbb{R}^n$ in (4) by $\mathbb{R}_+^n$. Moreover, using the fact that the rationals of the form $p/2^q$ $(p, q \in \mathbb{Z}_+)$ are dense everywhere in $\mathbb{R}_+$, one can write

$$g(w) = \inf_{q \in \mathbb{Z}_+} \min\{r_1(2^{-q}x_1) + \ldots + r_m(2^{-q}x_m) : \quad x_1, \ldots, x_m \in \mathbb{Z}_+^n, \quad (5)$$

$$2^{-q}x_1 + \ldots + 2^{-q}x_m = w\} = \inf_{q \in \mathbb{Z}_+} \left\{ 2^{-q} \min\{r_1(x_1) + \ldots + r_m(x_m) : \right.$$

$$\left. x_1, \ldots, x_m \in \mathbb{Z}_+^n, \quad x_1 + \ldots + x_m = 2^q w\} \right\} = \inf_{q \in \mathbb{Z}_+} 2^{-q} f(2^q w) .$$

Now let us say that $f(2w)$ has an *even realization* if there exist even vectors $a_1, \ldots, a_m \in \mathbb{Z}_+^n$ such that $a_1 + \ldots + a_m = 2w$ and $f(2w) = r_1(a_1) + \ldots + r_m(a_m)$. Let $w \in \mathbb{Z}_+^n$ be a vector for which $f(2w_0)$ does not have an even realization and the sum of the components of w_0 is as small as possible. As $f(0)$ has an even realization (namely $a_1 = \ldots = a_m = 0$), w_0 must have a strictly positive component, say the i^{th} one. Then, by the definition of w_0, $f(2w_0 - 2e_i)$ has an even realization, say $a_1, \ldots, a_m \in \mathbb{Z}_+^n$ (e_i is the i^{th} unit vector). As $f(2w_0 - 2e_i)$ is even by assumption and the definition of f implies $f(2w_0 - 2e_i) \geq f(2w_0) - 2$, therefore, either $f(2w_0 - 2e_i) = f(2w_0) - 2$ or $f(2w_0 - 2e_i) = f(2w_0)$. If the first one is the case then by adding $2e_i$ to s_i we obtain an even realization of $f(2w_0)$, contradicting to the definition of w_0. So $f(2w_0 - 2e_i) = f(2w_0)$ must hold. On the other hand, the existence of an even realization of $f(2w_0 - 2e_i)$ implies $f(2w_0 - 2e_i) = 2f(w_0 - e_i)$, which provides again an even realization for $f(2w_0)$, simply by doubling the (arbitrary) realization of $f(w_0 - e_i)$. That is, in any case we arrive at a contradiction to the definition of w_0, which shows that $f(2w)$ always has an even realization, implying $f(2w) = 2f(w)$. Continuing this by induction, we have

$$f(2^q w) = 2^q f(w) \quad (q \in \mathbb{Z}_+, \quad w \in \mathbb{Z}_+^n). \tag{6}$$

The combination of (5) and (6) yields $g(w) \equiv f(w)$.

Now we can continue by using a result of convex analysis. Let $C_1, \ldots, C_m \in \mathbb{R}_+^n$ be convex sets with support functions

$$h(a|C_i) = \sup\{ax : x \in C_i\} \quad (a \in \mathbb{R}^n),$$

respectively. Assume that $C_1 \cap \ldots \cap C_m \neq \emptyset$. Then

$$h(a|C_1 \cap \ldots \cap C_m) = \tag{7}$$
$$= \inf\{h(a_1|C_1) + \ldots + h(a_m|C_m) : \ a_1, \ldots, a_m \in \mathbb{R}^n, \ a_1 + \ldots + a_m = a\},$$

where $h(a|C_1 \cap \ldots \cap C_m)$ is the support function of $C_1 \cap \ldots \cap C_m$ (see Rockafellar [6], Corollary 16.4.1). Using $r_i(y) = h(a|P_i)$ and the definition (4) of g, (7) yields

$$h(w|P) = g(w) \qquad (w \in \mathbb{Z}_+^n), \tag{8}$$

where $P = P_1 \cap \ldots \cap P_m$. As we have proved that $g(w) \equiv f(w)$ $(w \in \mathbb{Z}_+^n)$, (8) implies

$$h(w|P) = f(w) \qquad (w \in \mathbb{Z}_+^n). \tag{9}$$

On the other hand, it follows from the definition of P that if w has negative components, then

$$h(w|P) = h(w^+|P) \qquad (w \in \mathbb{Z}_+^n), \tag{10}$$

where w^+ is obtained from w by changing the negative components to 0. So (9) and (10) together guarantee that the maximum value of any linear objective function with integral coefficients over P is an integer, which implies by the Edmonds-Giles Theorem [2] (see also Hoffman [4]) that P has integral vertices. Thus, P is the convex hull of the *common* independent sets of $\mathcal{F}_1, \ldots, \mathcal{F}_m$ and

$$h(w|P) = \max\{w(A): \ A \in \mathcal{F}_1 \cap \ldots \cap \mathcal{F}_m\} = f(w) \qquad (w \in \mathbb{Z}_+^n),$$

which completes the proof. ■

Now let us turn to the algorithmic consequences:

Theorem 2. *Assume that the family $(\mathcal{F}_1, \ldots \mathcal{F}_m)$ of independence systems has the minimax property and an oracle is given, which computes the sum $s(w) = r_1(w_1) + \ldots + r_m(w_m)$ of the weighted rank functions on input $w = (w_1, \ldots, w_m) \in \mathbb{R}_+^{mn}$. Then, using this oracle, the value of*

$$\max\{w(A): \ A \in \mathcal{F}_1 \cap \ldots \cap \mathcal{F}_m\} \qquad (w \in \mathbb{Z}_+^n)$$

can be computed in polynomial time.

Proof. As r_i is a convex function $(i = 1, \ldots, m)$, therefore, $s(x) = r_1(x_1) + \ldots + r_m(x_m)$ $(x = (x_1, \ldots, x_m))$ is also a convex function on $\mathbb{R}_+^{mn}$. Then the evaluation of $g(w)$ as defined in (4) requires the minimization of a convex function on an affine subspace of $\mathbb{R}_+^{mn}$. This problem is known to be solvable in polynomial time (see Yudin-Nemirovskii [5]). On the other hand, $g(w) = f(w)$ $(w \in \mathbb{Z}_+^n)$ as it has been shown in the previous proof, so Theorem 1 implies

$$g(w) = \max\{w(A): \ a \in \mathcal{F}_1 \cap \ldots \cap \mathcal{F}_m\}. \quad ■$$

Corollary. *If the family* $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ *of independence systems has the minimax property and the weighted rank function of each* $\mathcal{F}_i$ $(i = 1, \ldots, m)$ *can be computed in polinomial time, then*

$$\max\{w(A) : A \in \mathcal{F}_1 \cap \ldots \cap \mathcal{F}_m\} \qquad (w \in \mathbb{Z}_+^n)$$

can be computed in polynomial time. ■

4. Concluding remarks

Although Theorem 1 gives a necessary and sufficient condition for any family of independence systems to have the minimax property, it may not be easier to check this condition than the definition itself. On the other hand, as the equivalence is nontrivial, it may give some insight to the structure of independence systems. For example, it can be shown that the condition of Theorem 1 is satisfied in the case of two matroids, providing a new proof of the Matroid Intersection Theorem. Unfortunately, an important question remains open, however: are there further "good" examples beyond the case of two matroids?

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