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Projective Spaces and Colorings of $K_m \times K_n$

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ABSTRACT

Associated with Shelah's proof of van der Waerden's theorem are edge colorings of $K_m \times K_n$, where no $K_2 \times K_2$ is colored with two alternating colors. Using projective spaces, such colorings are obtained from $m = n = r^3$; $m = r^{2r-1}$, $n = 2r$; $m = r^{r+1}$, $n = r^2$ where r is the number of colors. It is shown that such an r-coloring exists for $m = n = r^{t+1}$ if the t-dimensional projective space of order r has the following property: there is a permutation of the points which maps each hyperplane into the complement of some hyperplane.

1. Introduction

Associated with Shelah's nice new proof of the van der Waerden's theorem, the following question is raised ([2], [3]). A "square" is a 4-cycle, that is a $K_2 \times K_2 \subset K_m \times K_n$. A *good r-coloring* of $K_m \times K_n$ is an r-coloring of the edges of $K_m \times K_n$ such that there are no squares with opposite edges having the same color. How large can n be, as a function of r, if $K_n \times K_n$ has a good r-coloring? Let $f(r)$ denote the largest such n. In particular, is $f(r) \leq r^c$ for some fixed constant c? It was shown in [3] that $f(r) \geq r^2$

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(with a simple proof); moreover, $f(r) \geq kr^3$ for some positive k (with a complicated proof). It was also mentioned that $n \geq r^3$ when r is a prime.

Here a simpler construction is given to show $f(r) \geq r^3$ for r a prime power. The method uses a property of projective planes, and will give $f(r) \geq r^{t+1}$ if the t-dimensional projective space of order r has this (complementary) property.

Let $PG(t,r)$ denote the t-dimensional projective space of order r over the finite field $GF(r)$ (r is a prime power). Call $H = PG(t,r)$ *complementary* if

(i) there exists a permutation π on the points of H such that the image of any hyperplane under π is disjoint from some hyperplane of H.

We shall use an equivalent definition of the complementary property. Since $PG(t,r)$ is self-dual, (i) is equivalent to

(ii) there exists a one-to-one map between the points and hyperplanes of H such that the image of any set of hyperplanes containing a given point is disjoint from some hyperplane of H.

Using Singer's theorem, $PG(t,r)$ can be generated by a difference set. The following property is stronger than (i):

(iii) there exist two disjoint difference sets generating H (like $\{1,2,4\}$ and $\{3,5,6\}$ for $PG(2,2)$).

Our interest in this complementary property comes from the following result.

Theorem 1. *If $PG(t,r)$ is complementary, then $f(r) \geq r^{t+1}$.*

For the case $t = 2$, the following can be proved.

Theorem 2. *The projective plane $PG(2,r)$ is complementary.*

Actually the proof of Theorem 2 works for arbitrary projective planes of order r.

From Theorems 1 and 2 we immediately get the following result.

Theorem 3. *If r is a power of a prime, then $f(r) \geq r^3$.*

Theorem 1 would imply that $f(r) \geq r^t$ for some t (at least for r a prime power) if the answer is affirmative to the following problem. (We believe that a negative answer would also be interesting.)

Problem. *Is there any $t \geq 3$ for which $PG(t,r)$ is complementary?*

As Aart Blokhuis [1] remarked, $t \leq r$ is necessary for the complementary property. Suppose to the contrary that $t > r$ and consider the inverse mapping π^{-1}. For a line L of $PG(t,r)$ the inverse image $\pi^{-1}(L)$ is a set of $r+1$ points. As $t > r$ this pointset is contained in a hyperplane H and since $\pi(H)$ contains the line L which intersects every hyperplane, $PG(t,r)$ cannot be complementary.

It is easy to show that $f(r) \leq r^{\binom{r+1}{2}}$ (see [3]). Thus, by Theorem 3, $f(2) = 8$. For the off-diagonal version of Shelah's problem, it is obvious that $K_{r^{\binom{r+1}{2}}} \times K_{r+1}$ has a good coloring. This is a consequence of coloring each of the $r^{\binom{r+1}{2}}$ copies of K_{r+1} differently, and then coloring the $r+1$ edges between two copies of a K_{r+1} such that the one pair of edges with the same color are in a square that is already well colored. From this, it is also obvious that $f(r) \geq r+1$. We have the following two results.

Theorem 4. *$K_{r^{2r-1}} \times K_{2r}$ has a good r-coloring.*

Theorem 5. *$K_{r^{r+1}} \times K_{r^2}$ has a good r-coloring if a projective plane of order r exists.*

Theorem 5 generalizes the result of [3] that $K_{r^2} \times K_{r^2}$ has a good r-coloring. The proofs are based on an equivalent formulation of a good r-coloring of $K_m \times K_n$.

An *(m,r)-coloring* is a coloring of the edges of a complete graph K with vectors of length m whose coordinates are from the set $\{1, 2, \cdots, r\}$. Let G_{ij} be the subgraph of K determined by those edges whose coloring vectors agree on the i^{th} and j^{th} coordinates. An (m,r) *coloring of K_n is good* if $\chi(G_{ij}) \leq r$ for each i and j with $1 \leq i < j \leq m$, where χ denotes the chromatic number of a graph.

Proposition 6. *An (m,r)-coloring of K_n is good if and only if $K_m \times K_n$ has a good r-coloring.*

Proof. Denote by $K_n^i = \{x_1^i, x_2^i, \cdots, x_n^i\}$ the i^{th} copy of K_n in $K_m \times K_n$ for $i = 1, 2, \cdots m$. Then, r-colorings of $\cup_{i=1}^m K_n^i \subset K_m \times K_n$ are in one-to-one correspondence with (m,r)-colorings of $K_n = \{y_1, y_2, \cdots, y_n\}$: the color of an edge e of K_n^i is identified with the i^{th} coordinate of the color vector of e. This correspondence is called the canonical mapping.

Assume a good (m,r)-coloring of K_n is given. Since $\chi(G_{ij}) \le r$, the vertices of G_{ij} have a proper r-coloring χ_{ij}. Color $K_n^i \subset K_m \times K_n$ according to the canonical mapping. The edges (x_p^i, x_p^j) of $K_m \times K_n$ between K_n^i and K_n^j is colored with $\chi_{ij}(y_p)$ for $p = 1, 2, \cdots, m$. Let $S = \{x_p^i, x_p^j, x_q^j, x_q^i\}$ be a square in $K_m \times K_n$. If $(x_p^i, x_q^i) = e_1$ and $(x_p^j, x_q^j) = e_2$ have different colors, then S is well colored. If e_1 and e_2 are colored with the same color, then the i^{th} and j^{th} coordinates of the coloring vector of $y_p y_q$ are the same (i.e. $y_p y_q \in E(G_{ij})$). Therefore, χ_{ij} colors y_p and y_q with different colors, and the colors of (x_p^i, x_p^j) and (x_q^i, x_q^j) are different. Thus $K_m \times K_n$ has a good r-coloring.

If $K_m \times K_n$ has a good r-coloring, then it implies an (m,r)-coloring on K_n by the canonical mapping. Then define χ_{ij} by coloring the vertex y_p of G_{ij} with the color of (x_p^i, x_p^j) in $K_m \times K_n$ for $p = 1, 2, \cdots, n$. Assume that $y_p y_q$ is an edge of G_{ij}. Then (x_p^i, x_q^i) and (x_p^j, x_q^j) have the same color in $K_m \times K_n$. Therefore, the square $S = \{x_p^i, x_p^j, x_q^j, x_q^i\}$ has different colors on the edges (x_p^i, x_p^j) and (x_q^i, x_q^j). Thus χ_{ij} assigns different colors to y_p and y_q, proving that $\chi(G_{ij}) \le r$, (i.e. K_n has a good (m,r)-coloring). This completes the proof of Proposition 6. ■

2. Proofs

In the spirit of Proposition 6, good (m,r)-colorings will be constructed in the following proofs.

Proof of Theorem 1. Consider the projective space $H_{t+1} = PG(t+1, r)$ and a hyperplane $H_t = PG(t,r)$ in H_{t+1}. Then $A_{t+1} = H_{t+1} - H_t$ is the $(t+1)$-dimensional affine space of order r. Let $a_1, a_2, \cdots, a_{r^{t+1}}$ denote the points of A_{t+1}. Identify the points of A_{t+1} with the vertices of a complete graph $K = K_{r^{t+1}}$.

A good (r^{t+1}, r)-coloring of K is defined as follows. The coloring vectors are associated with the hyperplanes of H_t. If h is a hyperplane of H_t, then h defines a partition $P(h)$ of A_{t+1} into r parallel hyperplanes, $B_1, B_2, \cdots, B_r$. Let $v(h) = [Y_1, Y_2, \cdots, Y_{r^{t+1}}]$ be the coloring vector defined by

$$Y_i = j \quad \text{if and only if} \quad a_i \in B_j.$$

Since H_t is complementary, there exists a one-to-one map α between points and hyperplanes of H_t with property (ii). Let (a_i, a_j) be an edge of K.

Consider the line $L(a_i, a_j)$ through a_i and a_j in H_{t+1}. Let $z = L(a_i, a_j) \cap H_t$, and let h be the hyperplane of H_t with $\alpha(h) = z$. Now (a_i, a_j) is colored with the vector $v(h)$.

To see that the given (r^{t+1}, r)-coloring is good, consider the set of coloring vectors V_{ij} which agree in the i^{th} and j^{th} coordinates. We have to show that the set of edges of K colored by vectors of V_{ij} determines an r-colorable graph G_{ij}. Let H_{ij} denote the set of hyperplanes of H_t satisfying $v(h) \in V_{ij}$. If an edge (a_p, a_q) of K is colored by $v(h) \in V_{ij}$, then a_p and a_q are in the same partition class of $P(h)$. Therefore, $L(a_p, a_q) \cap H_t = z \in h$ for each $h \in H_{ij}$. Thus, the hyperplanes of H_{ij} all contain z and then, by property (ii), $\alpha(H_{ij}) \cap h_{ij} = \emptyset$ for some hyperplane h_{ij} of H_t. If (a_p, a_q) is an edge of K colored with $v(h) \in V_{ij}$, then the definition of the color assignment implies $L(a_p, a_q) \cap H_t \in \alpha(H_{ij})$. Since $h_{ij} \cap \alpha(H_{ij}) = \emptyset$, the partition $P(h_{ij})$ on A_{t+1} defines a partition of K such that the set of edges colored by $v(h) \in H_{ij}$ are all between different classes of $P(h_{ij})$. Thus $\chi(G_{ij}) \leq r$ and Theorem 1 is proved. ■

Proof of Theorem 2. It is well known (and easy to prove by counting) that if H is a set of $r+1$ points in $PG(2,r)$, (or in any finite plane of order r) and H is not a line, then there exists a line H' such that $H \cap H' = \emptyset$. Therefore, to prove that $PG(2,r)$ is complementary, it is enough to find a permutation π that maps lines into non-lines. The number of permutations mapping a fixed line onto another fixed line is $(r+1)!(r^2!)$. Thus, the total number of wrong permutations is at most $(r^2+r+1)^2(r+1)!r^2!$, and it is easy to see that for $r \geq 3$ this number is smaller than $(r^2+r+1)!$, the total number of permutations on $PG(2,r)$. For the case $r = 2$, one can use (iii) noting that $\{1,2,4\}$ and $\{3,5,6\}$ are two disjoint difference sets generating the Fano plane. This completes the proof of Theorem 2. ■

The above proof works for arbitrary projective planes. For $PG(2,r)$ one can also use that it is possible to find two disjoint difference sets by taking a multiple of a difference set by a non-multiplier and using the same remark as in the beginning of the proof of Theorem 2. For example -1 is never a multiplier.

Proof of Theorem 4. It is easy to see that the edge set of $K_{r^{2r-1}}$ can be partitioned into $2r-1$ graphs $H_1, H_2, \cdots, H_{2r-1}$ so that each G_i is r-chromatic. The coloring vectors are defined as follows. The complete graph K_{2r} on vertices $\{x_1, x_2, \cdots, x_{2r}\}$ is factorized into 1-factors $F_1, F_2, \cdots, F_{2r-1}$. Then, each F_i defines a coloring vector of length $2r$ by setting the pairs to $1, 2, \cdots, r$ on the coordinate pairs corresponding to F_i. The $(2r, r)$-coloring of $K_{r^{2r-1}}$ is good since G_{ij} is just one of the graphs H_j. This completes the proof of Theorem 4. ■

Proof of Theorem 5. This proof is very similar to the proof of Theorem 4. A good (r^2, r)-coloring is defined on $K_{r^{r+1}}$ by partitioning the edges into $r+1$ graphs $H_1, H_2, \cdots H_{r+1}$ that are each r-chromatic. Then, each H_i is colored with a different vector defined by the $r+1$ partitions of the affine plane of order r. This completes the proof of Theorem 5. ■

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