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Sumsets and Powers of 2

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1. Introduction

For a finite set of positive integers A let $|A|$ denote the cardinality of A, and let $l(A)$ denote the maximal element of A. In the proof of a conjecture of Erdős and Freud it was shown by Erdős and Freiman [4] that there exists a constant l_0 such that for any A with $l(A) > l_0$ and $|A| > \frac{l(A)}{3}$, some power of 2 can be represented as a sum of distinct elements of A.

Nathanson and Sárközy [7] obtained an upper bound for the number of summands which is sufficient for such a representation. Namely, for a positive integer h let hA denote the set of all sums of h elements of A and let $h^\wedge A$ denote the set of all sums of h distinct elements of A. The result states that there exists an l_0 such that for any A with $l(A) > l_0$ and $|A| > \frac{l(A)}{3}$ there exists $h < 3504$ (resp. $h < 30361$) such that the set hA (resp. $h^\wedge A$) contains a power of 2.

In this paper we improve these results and prove the following two theorems.

Theorem 1. *For any finite set of positive integers A with $|A| > \frac{l(A)}{3}$, the set hA contains a power of 2 for some $h \le 6$.*

We note that in this result there is no any restriction to the number $l(A)$.

Theorem 2. *For any finite set of positive integers A such that*

$$l(A) > 42000 \tag{1}$$

and $|A| > \frac{l(A)}{3}$ the set $h^\wedge A$ contains a power of 2 for some $h \le 16$.

2. Long arithmetic progressions in the sets hK

Nathanson and Sárközy's proof of their result is based on Dyson's theorem ([3]). The main ingredient in the proof of our result is the following theorem from [6, page 11].

Theorem 3. *Let K be a finite set of k integers. If $|2K| = 2k-1+b$, where $0 \le b \le k-3$, then K is contained in an arithmetic progression of length $k+b$.* ∎

From now on till the end of the paper we work with a finite set of positive integers A and the set $K = A \cup \{0\}$. One has $l = l(K) = l(A), k = |K| = |A| + 1$ and $hK = \cup_{s=0}^{h} sA$,where $0A = \{0\}$. Let $d(K)$ denote the greatest common divisor of elements of K.

Corollary 1. *If $l \ge 2k-3$ and $d(K) = 1$, then $|2K| \ge 3k-3$.* ∎

Corollary 2. *If $l < 2k-3$, then $|2K| \ge k+l$.* ∎

Denote $[m, n] = \{x \in \mathbb{Z} | m \le x \le n\}$.

Theorem 4. *Let $l \le 2k-3$. Then*

$$[2l-2k+2, 2k-2] \subset 2K.$$

Proof. For any $z \in [l, 2k-2]$ the sets K and $z-K$ are parts of $[0, 2k-2]$. Thus, since both are of cardinalitity k, their intersection is not empty and $[l, 2k-2] \subset 2K$.

For any $z \in [2l-2k+2, l-1]$ the segment $[0, z]$ contains at least $k-(l-z)$ elements of each of the sets K and $z-K$. These sets have a nonempty intersection if $2(k-l+z) > z+1$ or $z > 2l-2k+1$. This proves that $[2l-2k+2, l-1] \subset 2K$. ∎

Theorem 5. *Let $d(K) = 1$ and $k > \frac{l}{3} + 1$. Then*

$$[2l + 2, 6l - 2] \subset 8K.$$

Proof. If $k \le \frac{l+3}{2}$, then $l \ge 2k - 3$. Using the conditions $d(K) = 1$ and $k > \frac{l}{3} + 1$ and Corollary 1 we obtain

$$|2K| \ge 3k - 3 \ge l + 1. \tag{2}$$

If $k > \frac{l+3}{2}$, then $|2K| \ge 2k - 1 > l + 1$. Thus, the inequality (2) is always fulfilled.

Now we apply either Corollary 1 or Corollary 2 to the set $2K$. In the case of Corollary 1 one has

$$|4K| \ge 3|2K| - 3 \ge 3l. \tag{3}$$

In the case of Corollary 2 one has $|4K| \ge l + 1 + 2l = 3l + 1$. Thus, the inequality (3) is always fulfilled.

We use Theorem 4 for the set $4K$, for which we already know that (3) holds and $l(4K) = 4l$. ■

The fact that the sets hK and $h^\wedge K$ contain a long arithmetic progression is very important for various applications (see, for example, [5] and [2]).

Corollary. *Theorem 1 is true for some $h \le 8$.*

Proof. Suppose first that $d(A) = 1$; then $d(K) = 1$ and $k > \frac{l}{3} + 1$. By Theorem 5 there exists an r such that $2^r \in 8K = \bigcup_{s=0}^{8} sA$.

Suppose now that $d(A) = 2$. For the set $\frac{K}{2} = \{x | x = \frac{a}{2}, a \in K\}$ we have $d(\frac{K}{2}) = 1$. Denoting $l_1 = \frac{l}{2}$ and applying Theorem 5 to $\frac{K}{2}$ we obtain $[2l_1 + 2, 6l_1 - 2] \subset 8(\frac{K}{2})$. Thus, there exists s such that $2^{s-1} \in 8(\frac{K}{2})$ and therefore $2^s \in 8K$. Since $|A| > \frac{l(A)}{3}$, the case $d(A) \ge 3$ is impossible. ■

Proof of Theorem 1. We may assume that l is not a power of 2. Then there exists $s \in \mathbb{N}$ such that $2l < 2^s < 4l$. We may also assume that $2^{s-1} \notin 4K$ and $2^{s-1} \notin 2^s - 2K = \{2^s - b | b \in 2K\}$. Now consider the sets $4K$ and $2^s - 2K$ which both lie in the set $[0, 4l] \setminus \{2^{s-1}\}$ of cardinality $4l$. Using (2) and (3), we get $|2^s - 2K| \ge l + 1$ and $|4K| \ge 3l$. Thus our two sets $2^s - 2K$ and $4K$ have a nonempty intersection which means that $2^s \in 6K$. ■

We note that a generalization of Theorem 2 for a more general condition on the quotient $\frac{k}{l}$ may be obtained. We plan to do it in another paper.

2. The case of distinct summands

Proof of Theorem 2. Let $A = \{a_1, a_2 \dots a_{|A|}\}$, where $1 \leq a_1 < a_2 < \dots < a_{|A|} = l$. Let $M = \{(a_i, a_j) \in A \times A | i < j\}$ and $M(c) = \{(a_i, a_j) \in A \times A | 0 < j - i \leq c\}, c \in \mathbb{N}$. Let σ denote the sum mapping $M \to 2A$ and let σ_c denote the restriction of σ to $M(c)$. Our first aim is to obtain a lower bound for the cardinality of the set

$$A_1 = \{b \in 2A | \, |\sigma^{-1}(b)| \geq 15\}.$$

For this we use the evident inequality

$$|A_1| \geq \frac{\sigma_c^{-1}(A_1)}{\max\limits_{b \in 2A} |\sigma_c^{-1}(b)|} \geq \frac{|M(c)| - |\sigma^{-1}(2A \setminus A_1)|}{\max\limits_{b \in 2A} |\sigma_c^{-1}(b)|}.$$

We now estimate the numbers in the right-hand side of this inequality. The number of pairs (a_i, a_j) with $j - i = d > 0$ is equal to $|A| - d$ and therefore

$$|M(c)| = \sum_{d=1}^{c} (|A| - d) = c|A| - \frac{c(c+1)}{2}.$$

Since $|2A| \leq 2l$ we have

$$|\sigma^{-1}(2A \setminus A_1)| \leq 14 \cdot 2l = 28l.$$

Let $b \in 2A$ and $\sigma_c^{-1}(b) = \{(a_{i_1}, a_{j_1}) \dots (a_{i_s}, a_{j_s})\}$, where $i_1 < i_2 < \dots < i_s < j_s < \dots < j_1$. Since $c \geq j_1 - i_1 \geq 2s - 1$, then $|\sigma_c^{-1}(b)| = s \leq \frac{c+1}{2}$. Now using $|A| > \frac{l}{3}$ we obtain

$$|A_1| \geq \frac{c|A| - \frac{c(c+1)}{2} - 28l}{\frac{c+1}{2}} \geq c_1 l - c, \tag{4}$$

where $c_1 = \frac{2}{3} - \frac{56\frac{2}{3}}{c+1}$.

We choose $c = 1500$. Then

$$l > \frac{c}{c_1 - 0.5} \tag{5}$$

because the right-hand side is less than 20000, which is less than l by (1).

We claim that $d(A_1) \le 3$. Indeed, if $d(A_1) \ge 4$, then $|A_1| \le \frac{l(A_1)}{d(A_1)} \le \frac{2l}{4} = 0.5l$ and (4) implies $l \le \frac{c}{c_1 - 0.5}$, which contradicts (5).

Consider the case $d(A_1) = 3$. We want to apply Theorem 4 to the set $\frac{K_1}{3}$, where $K_1 = A_1 \cup \{0\}$. For this we have to verify the inequality $l(\frac{K_1}{3}) \le 2|K_1| - 3$ (note that $|\frac{K_1}{3}| = |K_1|$). We have $l(\frac{K_1}{3}) \le \frac{2l}{3}$ and from (4) we have $|K_1| \ge c_1 l - c + 1 = m + 1$, where we denote $m = c_1 l - c$. Using (1) we obtain $l(\frac{K_1}{3}) \le \frac{2l}{3} < 2m - 1 \le 2|K_1| - 3$.

Using Theorem 4 we obtain $[2l(\frac{K_1}{3}) - 2|K_1| + 2, 2|K_1| - 2] \subset 2(\frac{K_1}{3})$. Add to both sides of this relation the set $2\{0, l(\frac{K_1}{3})\}$. We get

$$\Delta = [2l(\frac{K_1}{3}) - 2|K_1| + 2, 2l(\frac{K_1}{3}) + 2|K_1| - 2] \subset 4(\frac{K_1}{3}). \tag{6}$$

Since $|A| > \frac{l(A)}{3}$, there exists $a_{i_0} \in A$ such that $a_{i_0} \equiv 1 (\mathrm{mod}\, 3)$ or $a_{i_0} \equiv 2 (\mathrm{mod}\, 3)$. Let $w \in \mathbb{N}$ be the maximal number satisfying $2^w \equiv a_{i_0} (\mathrm{mod}\, 3)$ and

$$\frac{2^{w-2} - a_{i_0}}{3} < 2l(\frac{K_1}{3}) - 2|K_1| + 2. \tag{7}$$

We claim that

$$\frac{2^w - a_{i_0}}{3} \le 2l(\frac{K_1}{3}) + 2|K_1| - 2.$$

Suppose that this is not true. Then from (7) we get $2^{w-2} > 4|K_1| - 4$. On the other hand, (7) gives $2^{w-2} < a_{i_0} + 6l(\frac{K_1}{3}) - 6|K_1| + 6$, and we have $4|K_1| - 4 < l + 6l(\frac{K_1}{3}) - 6|K_1| + 6$, that is, $10|K_1| < 6l(\frac{K_1}{3}) + l + 10$. Using the inequalities $l(\frac{K_1}{3}) \le \frac{2l}{3}$ and $|K_1| \ge m+1$, we obtain $10m + 10 < 6 \cdot \frac{2l}{3} + l + 10$, that is, $l < \frac{c_1}{c - 0.5}$ which contradicts (5).

Since w is the maximal number satisfying (7), we see that $\frac{2^w - a_{i_0}}{3} \in \Delta$ and then (6) gives $2^w \in a_{i_0} + 4K_1$. This means that we can find such numbers $b_i \in A_1, 1 \le i \le t \le 4$, that

$$2^w = a_{i_0} + b_1 + b_2 + \ldots + b_t. \tag{8}$$

Each $(a_i, a_j) \in \sigma^{-1}(b)$, $b \in A_1$, gives a representation $a_i + a_j = b$. Suppose that $(a_{i_1}, a_{i_2}) \in \sigma^{-1}(b_1), \ldots, (a_{i_{2s-1}}, a_{i_{2s}}) \in \sigma^{-1}(b_s)$, $1 \le s < t$, are such that all $2s + 1$ integers $a_{i_0}, a_{i_1}, \ldots, a_{i_{2s}}$ are different. Since $2s + 1 \le 7$,

$|\sigma^{-1}(b_{s+1})| \geq 15$, there exists $(a_{i_{2s+1}}, a_{i_{2s+2}}) \in \sigma^{-1}(b_{s+1})$ such that all the integers $a_{i_0}, a_{i_1}, \ldots a_{i_{2s+2}}$ are different. Thus, we obtain a representation of 2^w as a sum of at most 9 different elements of K, i.e., $2^w \in 9^\wedge K$.

Consider now the case $d(A_1) = 1$. For the set K_1 we have $l(K_1) \leq 2l$ and $|K_1| \geq m + 1$.

Lemma.
(i) If $l(K_1) \leq 2m - 2$, then

$$[2l(K_1) - 2m, 6l(K_1) + 2m] \subset 8K_1. \tag{9}$$

(ii) If $l(K_1) > 2m - 2$ and $l(2K_1) < 2|2K_1| - 3$, then

$$[4l(K_1) - 6m + 2, 4l(K_1) + 6m - 2] \subset 8K_1. \tag{10}$$

(iii) If $l(K_1) > 2m - 2$ and $l(2K_1) \geq 2|2K_1| - 3$, then

$$[16l - 18m + 8, 18m - 8] \subset 8K_1. \tag{11}$$

Proof. (i) Applying Corollary 2 to the set K_1 we find $|2K_1| \geq l(K_1) + m + 1$. Applying Corollary 2 to the set $2K_1$ we get $|4K_1| \geq l(2K_1) + |2K_1| \geq 3l(K_1) + m + 1$. We now apply Theorem 4 to the set $4K_1$. Since $l(4K_1) = 4l(K_1)$, we get (9).

Finally we remark that if $l(K_1) > 2m - 2$ then, by Corollaries 1 and 2, one has

$$|2K_1| \geq 3m. \tag{12}$$

(ii) We apply Corollary 2 to the set $2K_1$ and from (12) we get $|4K_1| \geq l(2K_1) + |2K_1| \geq 2l(K_1) + 3m$. From Theorem 4 we obtain (10).

(iii) We apply Corollary 1 to the set $2K_1$ and from (14) we obtain $|4K_1| \geq 3|2K_1| - 3 \geq 9m - 3$. To apply Theorem 4 to the set $4K_1$ we have to verify the condition of the theorem. In view of $l(4K_1) \leq 8l$ and $|4K_1| \geq 9m - 3$ this condition will be true assuming $8l < 18m - 9$, which follows from (1). Thus we obtained (11). ■

If for the two positive integers x and y the condition $2x < y$ holds, then there exists $w \in \mathbb{N}$ such that $2^w \in [x, y]$. We claim that all the intervals constructed in the Lemma are of the form $[x, y]$ with $2x < y$. This is evident for the interval in (9). For the interval in (10), the condition is equivalent to the inequality $4l(K_1) < 18m - 6$. Since $l(K_1) \leq 2l$ and $m = c_1 l - c$, the later inequality follows from (1). For the interval in (11) the condition $2(16l - 18m + 8) < 18m - 8$ is true by (1). Thus $2^w \in 8K_1$ for some $w \in \mathbb{N}$.

We now use the same reasoning as in the case $d(A_1) = 3$. Let $2^w = b_1 + b_2 + \ldots + b_t$, $1 \leq t \leq 8$, $b_i \in K_1, w \in \mathbb{N}$. Suppose that $(a_{i_1}, a_{i_2}) \in \sigma^{-1}(b_1), \ldots, (a_{i_{2s-1}}, a_{i_{2s}}) \in \sigma^{-1}(b_s)$, $1 \leq s < t$, are such that that all $2s$ integers $a_{i_1}, a_{i_2} \ldots a_{i_{2s}}$ are different.

Since $2s \leq 14$, $|\sigma^{-1}(b_{s+1})| \geq 15$, there exists $(a_{i_{2s+1}}, a_{i_{2s+2}}) \in \sigma^{-1}(b_{s+1})$ such that all the integers $a_{i_1}, a_{i_2} \ldots a_{i_{2s+2}}$ are different. Thus, we obtain a representation of 2^w as a sum of at most 16 different elements of K, i.e. $2^w \in 16^\wedge K$.

The case $d(A_2) = 2$ is treated in the same way as in the end of the proof of Corollary of Theorem 5.

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