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The Birth of a Mathematical Theory in British India

H. GROPP

1. Introduction

> [3,p.353] The interest of mathematicians in combinatorial problems, involving the arrangement of a finite number of things in sets or patterns, satisfying given conditions, can be traced back to at least as far as Euler (1782), who interested himself in the construction of Latin and Graeco-Latin squares. ......It was, however, only about the beginning of the second quarter of the present century that the importance of combinatorial problems, for the proper designing of biological experiments, began to be understood, mainly through the work of Prof. R. A. Fisher and his associates. ......The object of this paper is to study the combinatorial problem involved in the construction of a certain type of design, first introduced in experimental studies by F. Yates (1936), and called *Balanced Incomplete Block Design.*

This is an extract of the introduction of a paper of R. C. Bose [3], an Indian mathematician, who in 1939 wrote a paper on combinatorial structures which are called balanced incomplete block designs. The publication of this paper can be regarded as the beginning of a new theory called *design theory.*

I want to give a short description of how this new theory was created. Although no previously unknown facts will be mentioned in the following I think that it is necessary to collect these facts in one paper. It is written in such a way that only basic mathematical knowledge is required to understand it. Thus it should be interesting for many readers, not only for specialists in design theory. The paper is related to a talk given by the author in a conference about "Science and Empires" in the UNESCO Building in Paris in April 1990. That is the reason why especially the "Indian beginning" of design theory is emphasized. The reader who is more interested in the development of design theory throughout the last 50 years is referred to books (e.g. [1]) and especially to a recent paper of H. Lenz [16].

1.1. A bit of mathematics

It is not the aim of my paper to give an introduction in design theory. Only a few basic facts will be mentioned briefly in order to enable the reader to understand what the objects of design theory are. For more details I refer the reader to special books on design theory (e.g. [1]).

Definition: A design (or more exactly) a $t-(v,k,\lambda)$ design is a set of v elements together with a family of subsets of k elements each such that every t elements occur together in the same subset exactly λ times. The number of subsets is denoted by b.

The main question of design theory is whether for a given set of parameters t,v,k,λ there is a $t-(v,k,\lambda)$ design or not, and if there is one, how many different designs can be constructed.

Some subclasses of designs have special names. If $v=b$ the design is called symmetric. Designs with $\lambda=1$ are called Steiner systems. If $t=2$ the design is shortly denoted by (v,k,λ) design.

Example: A small but very important design is the following:

$(v=b=7,\ k=3,\ \lambda=1)$

124 , 235 , 346 , 457 , 156 , 267 , 137

Take any 2 elements of the set $\{1,2,3,4,5,6,7\}$. Then you will find them together in exactly one of the 7 subsets of 3 elements above.

1.2. The sections of this paper

In Section 2 some early roots of design theory are described, especially the work of T. P. Kirkman of 1847 which could also had founded this theory 90 years earlier. Unfortunately, his papers were unknown for a long time and nobody followed his ideas.

Section 3 gives a short description of the method of the design of experiments, a statistical theory, which stimulated design theory very much.

Section 4 is the main section of my paper. At first short biographies of those mathematicians are given who were mostly involved in the development of the new theory. These people of different origins met each other in British India. Out of this cooperation a new theory was "born". The first important paper [3] of design theory which combines different mathematical branches is discussed. Some further important papers written during the following years are mentioned. They show how the new theory was developed during its first years.

In Section 5 I try to give a short survey of the evolution of the theory during the last 50 years. My paper cannot give an exhaustive enumeration of everything which happened but will mention some highlights only.

2. Some early roots of design theory

2.1. The reverend T. P. Kirkman

In 1844 the *Lady's and Gentleman's Diary*, edited by W. S. B. Woolhouse, contained the following Prize Question.

> Determine the number of combinations that can be made of n symbols, p symbols in each; with this limitation, that no combination of q symbols which may appear in any one of them shall be repeated in any other.

This question contains the question of the existence of Steiner triple systems $S(2,3,n)$ (Take $p = 3$ and $q = 2$). In modern design-theoretic language a Steiner system $S(t,k,v)$ is a $t-(v,k,1)$ design. The answer was given by Rev. T. P. Kirkman, the rector of a small parish in Lancashire in England to the Literary and Philosophical Society of Manchester on 15 December 1846 and published in [15].

The solution was: There exist Steiner triple systems $S(2,3,n)$ if and only if n is of the form $n = 6s + 1$ or $n = 6s + 3$.

This was the first time that a general theorem on such combinatorial structures was proved which are called designs today. It was, however, not the beginning of design theory for several reasons. The work of Kirkman remained unknown for a long time. More details about Kirkman's life, his work and some of his successors will be published in a paper on the history of Steiner systems $S(2,3,13)$ [11].

All these papers were purely combinatorial. They did not belong to one of the established mathematical theories of those days like algebra, analysis, etc. and interest in those questions was very strongly dependant on a few people who obtained these first results. No possible applications were known, neither within nor outside of mathematics.

2.2. Projective and affine geometry

Some very prominent designs have been known for a long time although even the terminology of designs was not known. The finite analogs of projective and affine planes were constructed in geometry and algebra. These are examples of designs for special sets of parameters. If there is a finite field of order n there can be constructed a $2-(n^2+n+1, n+1, 1)$ design and a $2-(n^2, n, 1)$ design. Hence such designs exist for all n where n is the power of a prime number. It was already known that there is no such design for $n = 6$.

2.3. Some singular results

There are some other combinatorial results which, however, were not followed by the creation of a theory. Some biplanes, i.e. $2-(v, k, 2)$ designs, were constructed as spatial configurations by E. Kummer (1864) and V. Martinetti (1896) for $v = 16, k = 6$ and $v = 11, k = 5$ resp. In 1893 J. Hadamard [12] constructed Hadamard matrices of orders 12 and 20 which can be regarded as $2-(11,5,2)$ and $2-(19,9,4)$ designs.

There were two papers written independently from the development described in the following sections but at the same time (1938) which also had a big influence on the new theory. Both papers discuss the action of automorphism groups on designs and use this action to construct a lot of designs. The authors were J. Singer [21] in the United States and E. Witt [22] in Germany.

3. The design of experiments

3.1. A short description of the method

The method of experimental design was developed after the First World War at Rothamsted Experimental Station in Southern England. This method tries to find out how much an experiment depends on certain parameters. It looks for a possibility to get the correct result with a high statistical certainty without too many expensive and long trials. In a paper of F. Yates [23] of 1936 the following description is given.

> [23,p.121] INTRODUCTION. Most biological workers are probably by now familiar with the methods of experimental design known as randomized blocks and the Latin square. These were originally developed by Prof. R. A. Fisher, when Chief Statistician at Rothamsted Experimental Station, for use in agricultural field trials.
>
> [23,p.122] The present paper describes another possible modification of the randomized block type of arrangement. ...... In this modified type of arrangement the number of experimental units per block is fixed, being less than the number of treatments, and the treatments are so allotted to the blocks that every two treatments occur together in a block equally frequently. With six treatments a, b, c, d, e, f, and blocks of three experimental units, for example, the following grouping of treatments into ten blocks satisfies the above conditions:

1	a	b	c	6	b	c	f
2	a	b	d	7	b	d	e
3	a	c	e	8	b	e	f
4	a	d	f	9	c	d	e
5	a	e	f	10	c	d	f

> In this set of groupings every two treatments occur together in a block twice, there being five replications of each treatment.
>
> It is proposed to call this type of arrangement a *symmetrical incomplete randomized block arrangement*, or more briefly, when the symmetry and randomization are understood, an *incomplete block arrangement.*

The above example is a $2-(6,3,2)$ design with $b=10$. The paper by Yates is the first one on balanced incomplete block designs.

On page 127 Yates defines the symbols he uses: t for the number of treatments (today we use v which stands for variety), k for the number of experimental units per block, r for the number of replications of each treatment, b for the number of blocks and λ for the number of times any two treatments occur together in a block. Yates also gives the two main conditions which must hold if such a designs exists (Attention: t is called v now.)

$$tr = bk$$

and

$$\lambda(t-1) = r(k-1).$$

Table VIII in the paper of Yates [23] gives examples of designs explicitly, e.g. the designs (7,3,1), (11,5,2), (13,4,1), (16,6,2), (21,5,1). He mentions the connection of projective and affine planes with Latin squares. Yates also discusses designs which are known as biplanes today.

There is a collection of tables, edited by R. A. Fisher and F. Yates [9] which shows a lot of explicit solutions. It has been translated into many languages of the world and is actualized very frequently. The tables are written in such a way that they can be used directly for applications in biology, agriculture and medicine. A comparison of the different editions can give a good description of the progress in the design of experiments.

4. The meeting in India

4.1. Ronald Aylmer Fisher (1890–1962)

R. A. Fisher was born on February 17, 1890 in East Finchley (London). Fisher studied in Cambridge mathematics, physics and statistics. After the end of the First World War he became statistician at the Rothampsted Agricultural Station and built up a statistical laboratory there. In 1933 he became Galton Professor in the University of London and editor of the Annals of Eugenics. His interests were spread widely in mathematics, statistics and genetics. Fisher's voyage to India will be described below. This short biography cannot contain all important aspects of Fisher's scientific work. I refer the reader to numerous books about this work, e.g. [10]. Fisher's biography was written by his daughter Joan Fisher Box [5]. Fisher travelled a lot into many parts of the world. He spent his last years in Adelaide (Australia) and died there on July 29, 1962.

4.2. Friedrich Wilhelm Levi (1888-1966)

A further important contributor was the German F. W. Levi, born February 6, 1888 in Mülhausen (Elsass). Levi studied in Würzburg and Strassburg and continued his research in Göttingen and Leipzig. His dissertation *Körper und Integritätsbereiche 3. Grades* (1911) and his Habilitationsschrift *Abelsche Gruppen mit abzählbaren Elementen* (1919) show his interest in algebra. Between 1919 and 1935 he was Privatdozent and later professor of geometry in Leipzig.

In 1935 he was forced to leave Nazi Germany and emigrated to India. In Calcutta he became Head of the Department of Pure Mathematics. From 1948 till 1952 he was in Bombay in the Tata Institute of Fundamental Research. In 1952 he returned to Germany where he became professor in West Berlin and Freiburg. He died on January 1, 1966 in Freiburg. A short biography in German is contained in [18].

4.3. Raj Chandra Bose (1901-1987)

R. C. Bose was born in Hoshangabad (Madhya Pradesh, India) on June 19, 1901. Bose studied in Delhi and Calcutta. In 1933 he joined the Indian Statistical Institute in Calcutta, founded by P. C. Mahalanobis.

Bose's early interests in mathematics had been mainly geometrical. For his future work he had to learn statistics. In his short autobiography Bose reports what Mahalanobis told him.

> [4,p.87] You were saying that you do not know much statistics. You master the 50 papers, the list of which you have received, and Fisher's book. This will suffice for your statistical education for the present.

Bose should become the one who successfully combined the different mathematical disciplines and gave birth to the new theory [3].

Bose left India in 1947 and became professor at the University of North Carolina at Chapel Hill until 1971 and at Colorado State University in Fort Collins until 1980. In all these years he did a lot both for the progress of design theory as well as for the scientific contact between the United States and the independent Indian state. Bose died on October 31, 1987.

4.4. The first Indian Statistical Congress

After F. W. Levi had arrived in India he held seminars on algebra and geometry in the University of Calcutta which were attended by R. C. Bose. After 1938 Bose also taught modern algebra.

An important step towards the birth of design theory was the organization of a first conference of statistics in India. A quicker development of statistics in India was urgently needed for the evolution of agriculture and medicine in India. The founder of the Indian Statistical Institute, P. C. Mahalanobis who had studied in England invited his former teacher and one of the leading statisticians of the world, R. A. Fisher, to be the president of a special Statistical Conference which should be held together with the annual Indian Science Congress in 1938 in Bombay.

Fisher really came to India as president of the conference and gave 6 lectures at the University of Calcutta, too. He brought with him some problems in the design of statistical experiments. In his autobiography Bose described the situation as following.

> [4,p.89] He posed to us some problems arising out of his study. It occurred to me that I could successfully use the theory of finite fields and finite geometries, which I was then teaching, as tools for the construction of designs. I solved during his stay one of the problems he had posed. He asked me to develop my methods systematically and send him a paper which he promised to publish in the *Annals of Eugenics* of which he was editor. Thus my paper on "The Construction of Balanced Incomplete Designs" came to be written in 1939; it has now become a classic and is quoted in every book on the subject.

Indeed it happened that this important design-theoretical paper was published in a biological journal which in many cases is not available in mathematical libraries of today. It contains a lot of solutions of designs the existence of which was unknown at that time.

I refer the reader to the few words of Bose's introduction which I cited in the beginning of my paper. Now I want to give a short report on the contents of Bose's paper. After a reference to books and papers which had been written about geometry and combinatorics before Bose writes about the object of his paper:

> [3,p.356] In this paper I have attempted to make a systematic study of the problem of construction of balanced incomplete block designs. It has been possible to obtain a number of new solutions, thus filling in some gaps in the tables of Fisher & Yates (1938). ......The method of constructing such solutions is based on the concept of *Symmetrically repeated differences.* By the use of this concept it is possible to construct the whole solution with the help of a few initial blocks. The discovery of the initial blocks is much facilitated by the use of properties of primitive roots of binomial equations in Galois fields, but the use of these properties is not essential.

In fact, Bose uses algebraic concepts to construct a lot of unknown designs. In his first 3 paragraphs (p.357–p.372) he gives basic facts on algebraic structures like moduls, rings, and fields, he discusses designs which can be obtained from finite geometry and proves his 2 fundamental theorems on differences which are essential for the rest of the paper.

In the following (p.372–p.389) he gives general solutions for the cases (k, λ)=(3,1), (3,2), (4,1), and (5,1). In all cases he gives examples for certain values of v explicitly. The last paragraph (p.390–p.396) gives constructions of symmetric designs (i.e. $v = b$).

In a footnote to his summary Bose discusses the problem of finding all different designs with the same parameters.

> [3,p.396] Two combinatorial solutions for the same design may be said to be isomorphic, when they are identical except for a change of notation (i.e. renaming of varieties and blocks). The question whether solutions obtained in this paper are isomorphic to the corresponding solutions in Fisher & Yates' (1938) tables, has not been investigated. The problem of obtaining the totality of independent (non-isomorphic) combinatorial solutions for a given design, is of great theoretical as well as practical interest, and awaits study.

4.5. The first years of design theory

In the following I want to discuss some papers which were published during the following years. Many of them were written in India.

The same volume in which the paper by Bose was published contains a paper by S. R. Savur [19] with a note of the editor R. A. Fisher. He says:

> [19,p.45] EDITOR'S NOTE. The problem of finding solutions of problems in incomplete blocks has in recent times attracted the attention of mathematicians interested in the design of experiments. ...... The following note discusses this problem subject to the restrictions $k = 3, \lambda = 1$. ...... It is, however, not apparent that his method will always solve the problem when $v = 4m + 1$. R.A.F.

This remark shows that the results of Kirkman of 1847 were not known to Fisher and his colleagues. The above mentioned paper by Kirkman [15] solved the existence problem already totally for $k = 3$ and $\lambda = 1$. For more details see [11]. In 2 examples Savur deals with the cases $v = 15$ and $v = 13$ although in 1939 not only the existence was certain but also all these designs had been classified in 1919 by F. N. Cole, L. D. Cummings and H. S. White [6] and in 1899 by V. de Pasquale [17] resp. This shows that the "parents" were not aware of the early roots of their "newborn child".

The following volume of *Annals of Eugenics* contains a paper by R. A. Fisher himself [7]. He describes the "child" in his introduction.

> [7,p.52] Recent papers in the *Annals of Eugenics* by Yates (1936) and Bose (1939) have drawn attention to the importance of the combinatorial problem which arises when it is desired to compare a number of "varieties", or experimental treatments, on "blocks" of experimental material, which, for the sake of greater homogeneity contain fewer units than the number of varieties to be used. ......Bose, while adding further solutions to those so far discovered, has discussed the intimate connexion of this problem with other branches of mathematics, notably with finite geometries.

He continues as follows:

> [7,p.52] Although the greatest practical importance attaches to the first solution of such a problem, it is also of some theoretical interest to discover what other types of solution may exist. ......As a preliminary let us set out the primary arithmetical requirements, and demonstrate an important inequality.

This is the question for non-isomorphic solutions, many years after de Pasquale and Cole, Cummings, White had solved it. The mentioned inequality says $v \leq b$ and is now called the Fisher inequality.

In this paper Fisher determines the number of (15,3,1) designs as 79. The correct number, however, is 80 (compare [6] in 1919). In his summary Fisher says:

> [7,p.66] The paper reports the results of an exploration, and is of interest principally as a study in method. ......Methods of specifying the invariant characteristics of a solution will vary much with the size of the blocks, and even for blocks of 3 with the frequency (λ) with which each pair of varieties is tested together. ......Seventy-nine sets of solutions have been found for the particular problem of selecting 35 blocks of 3 out of 15 varieties, in such a way that each variety appears seven times and each pair of varieties once in the same block. It is difficult to assess the probability that any further set has escaped detection.

Although Fisher did not find the correct number of designs this paper was the first in a series of papers of similar contents.

In a further paper [8] of 1941/42 Fisher discussed the existence of special symmetric designs, namely (22,7,2), (29,8,2), (25,9,3), (31,10,3), and (46,10,2).

> [8,p.291] It seems extremely probable that no solution exists, at least for the first two of these cases.

In fact, the above designs no. 1, 2, and 5 do not exist while no. 3 and 4 exist. The nonexistence of designs no. 1 and 2 was proved by Q. M. Hussain in [14](1946) and [13](1945) resp. In 1949 M. P. Schützenberger [20] proved a general theorem which implies the non-existence of design no. 5. In 1944 K. N. Bhattacharya [2] constructed design no. 3.

These examples, I think, show the quick progress of the new theory as well as the big influence of Indian mathematicians like Bose, Hussain, and Bhattacharya.

5. The 50 years after the birth

Of course, it is not possible to give a detailed report on the development of design theory between 1940 and 1990 in the last section of this paper. I just want to give a few facts concerning the history and refer the reader to other papers on this subject. In order to keep the list of references relatively short the reader is kindly requested to look for the exact references of the following papers in books on design theory.

A first important result was an extension of the result of Schützenberger [20] by R. H. Bruck, H. J. Ryser, S. Chowla (1949/50) which implied a lot of non-existence results. Up to now this has been the only general result concerning non-existences of designs. After 1955 more and more non-statisticians joined the people who already worked in design theory. Especially relations to group theory, number theory and geometry were discussed. A lot of existence results for certain designs were obtained during these 50 years. Only in few cases all non-isomorphic designs with a certain parameter set have been determined.

The most discussed problem through all these years has been the question of the existence of a projective plane of order 10, i.e. a 2-(111,11,1) design. Such a structure cannot be constructed by algebraic means (10 is not the power of a prime) and its existence is not excluded by the result of Bruck, Ryser and Chowla. Very recently it was proved by C. Lam et al. (1988) that such a design cannot exist. He used a big amount of computing time and the truth of the result depends very much on this new method of proving mathematical results.

This is, however, only one new instrument which is used to obtain new results in design theory. Both the number of methods and the number of people involved in design theory is still increasing. Design theory is important today especially in North America, Israel, Australia, New Zealand, Japan, a lot of European countries and last but not least India which is still contributing an important part to the progress of design theory.

Since mathematics is a very international science it is not as important as in other sciences where new ideas are produced and new theories are "born". I think, however, it is very interesting to watch the development of design theory in the political and historical context of its time.

Probably design theory or something very similar would have been created somewhere else if some of the historical events mentioned above had not happened. On the other hand, there should be no doubt that it is not possible to really understand a theory without knowing something about its history and development.

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Harald Gropp

Mühlingstr. 19,
D-6900 Heidelberg, Germany