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Edge Disjoint Cliques in Graphs

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ABSTRACT

Two related concepts are studied: the decomposition of graphs into edge disjoint cliques of either given or arbitrary order and the number of edge disjoint cliques of given order. We survey the main results and open problems in the subjects.

1. Decompositions

The classical theorem of extremal graph theory about cliques is Turán's famous theorem.

Theorem 1. ([9] $r = 3$, [10] $r \geq 4$). *If G is a graph of n vertices not containing any clique (or complete subgraph) K_r of r vertices then*

$$|E(G)| \leq t_{r-1}(n),$$

where $t_{r-1}(n)$ is the edge number of the $(r-1)$-partite Turán graph $T_{r-1}(n)$ $(= \overline{K_{n_1}} + \overline{K_{n_2}} + \ldots + \overline{K_{n_{r-1}}},\ n_1 \leq n_2 \leq \ldots \leq n_{r-1} \leq n_1 + 1)$ and equality holds if and only if $G \simeq T_{r-1}(n)$.

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If we should like to decompose $T_2(n)$ into cliques then the only possibility is to decompose it into $t_2(n)$ K_2's. The first theorem on clique decompositions (proved by P. Erdős, A. W. Goodman and L. Pósa) states that this is the worst case.

Theorem 2. ([4]). *The edge set of any graph G of n vertices can be decomposed into at most $t_2(n)$ edge disjoint cliques and if G is not isomorphic to $T_2(n)$ then strictly fewer cliques suffice.*

A generalization of Theorem 2 was conjectured by G.O.H. Katona and T. Tarján and was proved independently by F.R.K. Chung [2], A.V. Kostochka and the present author [7].

Theorem 3. *For an arbitrary graph G, let $p(G)$ denote the minimum of $\sum |V(G_i)|$ over all decompositions of G into edge disjoint cliques $G_1, G_2, \dots$ (i.e. the subgraphs G_i are pairwise edge disjoint cliques such that $E(G) = \cup E(G_i)$). Then*

$$p(G) \leq 2t_2(n)$$

and equality holds if and only if $G \simeq T_2(n)$.

Notice that in Theorems 2 and 3, the decompositions can have cliques of any order. However, similar theorems can be proved for the decompositions of graphs into edge disjoint cliques K_r and K_2. Naturally, if we want to decompose the Turán graph $T_{r-1}(n)$ into edge disjoint K_r's and K_2's then the only way to do it is to decompose the graph into $t_{r-1}(n)$ K_2's. The following theorem states the Turán graph $T_{r-1}(n)$ yields the worst case for any $r \geq 3$. The case $r = 3$ was already settled in the paper of P. Erdős, A.W Goodman and L. Pósa [4], the cases $r \geq 4$ were proved by B. Bollobás [1].

Theorem 4. *The edge set of any graph of n vertices can be decomposed into at most $t_{r-1}(n)$ edge disjoint cliques of r or 2 vertices and $t_{r-1}(n)$ cliques are necessary if and only if $G \simeq T_{r-1}(n)$ $(r \geq 4)$, $G \simeq T_2(n)$, K_4 or K_5 $(r = 3)$.*

The following theorem is a sharpening of Theorem 4 (like Theorem 3 is a sharpening of Theorem 2) for the case $r \geq 4$ and was proved by Z. Tuza and the present author [8].

Theorem 5. *For an arbitrary graph G, let $p_r(G)$, $r \geq 4$ denote the minimum of $\sum |V(G_i)|$ over all decompositions of G into pairwise edge disjoint cliques $G_1, G_2, \dots$ of r or 2 vertices. Then*

$$p_r(G) \leq 2t_{r-1}(n)$$

and equality holds if and only if $G \simeq T_{r-1}(n)$.

It is a bit surprising that the desired inequality

$$p_3(G) \le 2t_2(n)$$

does not hold for all graphs G. It is easy to see that

$$p_3(K_5) = 14 = 2t_2(5) + 2$$

and

$$p_3(K_{6k+4}) = 2t_2(6k+4) + 1 .$$

In the case $r = 3$, the following estimate was proved only.

Theorem 6. ([8]). *For every graph G of n vertices,*

$$p_3(G) \le \frac{9}{16} n^2 .$$

However, we conjecture that much stronger estimate holds, as well.

Conjecture 1. *For every graph G of n vertices,*

$$p_3(G) \le 2t_2(n) + o(n^2) = \frac{1}{2} n^2 + o(n^2) .$$

Another form of this conjecture is

Conjecture 1'. *Every graph of n vertices and $t_2(n) + m$ edges contains $\frac{2}{3}m - o(n^2)$ edge disjoint triangles.*

A possible (maybe too optimistic) generalization of it is

Conjecture 2. *Every graph of n vertices and $t_{r-1}(n) + m$ edges contains $\frac{2}{r}m - o(n^2)$ edge disjoint cliques of r vertices.*

P. Erdős suggested to study a similar weighting function of the decompositions of graphs into edge disjoint cliques. For any graph G, let $p^*(G)$ denote the minimum of $\sum |V(G_i) - 1|$ over all decompositions of G into pairwise edge disjoint cliques $G_1, G_2, \ldots$. A maybe too optimistic conjecture is as follows:

Conjecture 3. *For every graph G of n vertices,*

$$p^*(G) \le t_2(n).$$

This conjecture seems to be just a bit stronger than Theorem 3, but it is not the case. To show the deepness of Conjecture 3, let me mention my favorite special case, which is still open, as well.

Conjecture 4. *Every K_4-free graph of n vertices and $t_2(n) + m$ edges contains m edge disjoint triangles.*

Only the following even weaker special case is settled.

Theorem 7. *Every 3-colorable graph of n vertices and $t_2(n) + m$ edges contains m edge disjoint triangles.*

2. On the number of edge disjoint cliques

With Theorem 5, 6, 7 and Conjectures 1', 2, 4, we arrived at problems related to the number of edge disjoint cliques of given order. The basic question is as follows:

How many pairwise edge disjoint cliques of r vertices can be found in a graph of n vertices and $t_{r-1}(n) + m$ edges?

Naturally, if we add m edges to the Turán graph $T_{r-1}(n)$ arbitrarily then the resulting graph contains at most m edge disjoint cliques of r vertices, and if m is large enough and the edges are added to the Turán graph $T_{r-1}(n)$ in a special way then it may occur that we cannot find m edge disjoint cliques of r vertices. Thus, it was natural that the first two problems on the number of edge disjoint cliques ask for what values of m can we find m or asymptotically m edge disjoint cliques of r vertices. Of course, Erdős [3] only proposed these problems for triangles first.

Problem. *Determine the maximum m such that every graph of n vertices and $t_2(n) + m$ edges has m edge disjoint triangles.*

Conjecture. *There is a function $f(c)$ such that every graph of n vertices and $t_2(n) + cn$ edges contains $cn - f(c)$ edge disjoint triangles.*

It turned out that these problems are very closely related to each other. Actually, the following generalization of Erdős' conjecture was fundamental in the solution of the problem.

Theorem 8. ([5] $r = 3$, [6] $r \geq 4$). *Let G be a graph of n vertices and $t_{r-1}(n) + m$ edges where $m = o(n^2)$. Then G contains*

$$m - O\left(m^2/n^2\right) = (1 - o(1))\,m$$

edge disjoint cliques of r vertices.

Using the case $r = 3$, we proved the following theorem answering the problem of Erdős.

Theorem 9. ([5]). *Let G be a graph of n vertices and $t_2(n) + m$ edges where*

$$m \leq 2n - 10 \quad \text{if } n \text{ is odd}$$
$$\text{and} \quad m \leq 1.5n - 5 \quad \text{if } n \text{ is even.}$$

Then the graph G contains m edge disjoint triangles if n is sufficiently large. Examples 1 and 2 show these upper bounds are sharp.

Example 1. $n = 2k$. $G = T_2(n-3) + K_3$, or in other form, this graph G is obtained from the Turán graph $T_2(n)$ by taking one vertex of the first color class and two vertices of the second color class and joining these 3 vertices to all vertices. Then it is easy to see that the resulting graph G of n vertices and $t_2(n) + 1.5n - 4$ edges does not contain $1.5n - 4$ edge disjoint triangles.

Example 2. $n = 2k + 1$. Take the Turán graph $T_2(n)$ with color classes V_1 and V_2 of k and $k+1$ vertices, respectively. Take one vertex in V_1 and join it to all vertices in V_1. Then, take three vertices, say x_1, x_2, x_3, in V_2, and join them to each other and to $k - 3$ (all except one) other vertices in V_2. (The sets of $k - 3$ vertices do not have to be the same for the vertices x_1, x_2, x_3.) It is easy to see that the resulting graph G of n vertices and $t_2(n) + 2n - 9$ edges does not contain $2n - 9$ edge disjoint triangles.

Remark 1. *Theorem 9 is sharp only for the graphs in Examples 1 and 2 if n is sufficiently large. Notice that the extremal graph is unique when n is even and we have three similar extremal graphs when n is odd.*

Remark 2. *In [5], in the not detailed case of odd n, we stated incorrectly that $m \leq 2n - 9$ is sufficient and the appropriate example given in [5] is not the best either.*

The case $r = 3$ of Theorem 8 was the most important tool in the proof of Theorem 9. Thus, it is not a surprise that when we managed to prove Theorem 8 for any $r \geq 4$ then we managed to generalize Theorem 9 for $r \geq 4$, as well.

Theorem 10. ([6]). *Let $r \geq 4$ and G be a graph of n vertices and $t_{r-1}(n) + m$ edges where*

$$m \leq 3 \left\lfloor \frac{n+1}{r-1} \right\rfloor - 5 .$$

Then G contains m edge disjoint cliques of r vertices if n is sufficiently large. Example 3 shows that this upper bound is sharp and this is the only extremal graph.

Example 3. Take two color classes V_i and V_j of $\left\lfloor \frac{n+1}{r-1} \right\rfloor$ vertices in $T_{r-1}(n)$. (I.e. take two color classes of $\left\lfloor \frac{n}{r-1} \right\rfloor$ vertices if there are at least two of this size, and take two color classes of $\lceil \frac{n}{r-1} \rceil$ vertices otherwise.) Add $3 \left\lfloor \frac{n+1}{r-1} \right\rfloor - 4$ edges to $T_{r-1}(n)$ so that two elements of V_i and one element of V_j should be joined to all vertices. It is easy to see that the resulting graph of n vertices and $t_{r-1}(n) + 3 \left\lfloor \frac{n+1}{r-1} \right\rfloor - 4$ edges does not contain $3 \left\lfloor \frac{n+1}{r-1} \right\rfloor - 4$ edge disjoint cliques of r vertices since the subgraph induced by $V_i \cup V_j$ does not contain $3 \left\lfloor \frac{n+1}{r-1} \right\rfloor - 4$ edge disjoint triangles (see Example 1).

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