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On Arithmetic Graphs Associated with Integral Domains, II

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1. Introduction

In [6], [7] and [8], some quantitative finiteness results have been established for arithmetic graphs associated with integral domains. These led to important applications concerning diophantine equations, algebraic number theory, irreducible polynomials and pairs of polynomials with given resultant, respectively (see [6], [7], [8] and the references given there). The purpose of this paper is to give a new and improved version of Theorem 1 of Part I [8]. This enables us to get much better bounds (in terms of certain important parameters) in applications to irreducible polynomials (cf. [10]) and to pairs of polynomials with given resultant or given semi-resultant (cf. [9]).

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2. Results

We keep the notation of Part I. Let K be a finitely generated extension field of $\mathbb{Q}$, R a subring of K containing 1, U a finitely generated subgroup of the unit group of R with $-1 \in U$, and N a finite non-empty subset of $R \setminus \{0\}$. For each pair of distinct positive integers i, j we select an element of N, denoted by $\delta_{i,j}$, such that $\delta_{i,j} = \delta_{j,i}$. For any finite ordered subset $A = \{\alpha_1, \dots, \alpha_m\}$ of R, we denote by $\mathcal{G}(A)$ the simple graph with vertex set A whose edges are the (unordered) pairs $[\alpha_i, \alpha_j]$ for which

$$\alpha_i - \alpha_j \notin \delta_{i,j} \cdot U.$$

The ordered subsets $A = \{\alpha_1, \dots, \alpha_m\}$ and $A' = \{\alpha'_1, \dots, \alpha'_m\}$ of R are called *U-equivalent* if $\alpha'_i = \varepsilon\alpha_i + \beta$ for some $\varepsilon \in U$ and $\beta \in R$, $i = 1, \cdots, m$. It is obvious that in this case the graphs $\mathcal{G}(A)$ and $\mathcal{G}(A')$ are isomorphic.

Denote by $\overline{\mathcal{G}(A)}$ the complement of $\mathcal{G}(A)$, and by $\overline{\mathcal{G}(A)}^{\circ}$ the *polygon hypergraph* of $\overline{\mathcal{G}(A)}$, i.e. that hypergraph whose vertices are the edges of $\overline{\mathcal{G}(A)}$ and whose edges are the cycles[1] $\alpha_{i_1}, \dots, \alpha_{i_k}$ $(k \geq 3)$ in $\overline{\mathcal{G}(A)}$ such that

$$\sum_{j \in J} (\alpha_{i_j} - \alpha_{i_{j+1}}) \neq 0 \text{ for each non-empty subset } J \text{ of } \{1, \dots, k-1\}. \quad (1)$$

If, in particular, we consider only cycles of length 3 (i.e. triangles) of $\overline{\mathcal{G}(A)}$ the hypergraph so obtained is called the *triangle hypergraph* of $\overline{\mathcal{G}(A)}$ (cf. [1], p. 440) and is denoted by $\overline{\mathcal{G}(A)}^{\triangle}$ (cf. [6], [7]). Similarly, if we take only cycles of length 3 and 4 of $\overline{\mathcal{G}(A)}$ then the corresponding hypergraph, denoted by $\overline{\mathcal{G}(A)}^{\square}$, is called the *quadrangle hypergraph* of $\overline{\mathcal{G}(A)}$.

Let $\delta_0, \delta_1, \dots, \delta_n$ be non-zero elements of K, and consider the unit equation

$$\delta_0 x_0 + \delta_1 x_1 + \dots + \delta_n x_n = 0 \quad \text{in} \quad x_0, x_1, \dots, x_n \in U. \quad (2)$$

If $\mathbf{x} = (x_0, \dots, x_n)$ is a solution of (2) then so is $\varepsilon\mathbf{x} = (\varepsilon x_0, \dots, \varepsilon x_n)$ for every $\varepsilon \in U$. Such solutions are proportional to each other. A solution $\mathbf{x}$

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[1] In other words, $\alpha_{i_1}, \dots, \alpha_{i_k}$ $(k \geq 3)$ are distinct elements of A such that $[\alpha_{i_1}, \alpha_{i_2}], \dots, [\alpha_{i_{k-1}}, \alpha_{i_k}][\alpha_{i_k}, \alpha_{i_1}]$ are all edges in $\overline{\mathcal{G}(A)}$ (cf. [2]).

of (2) is called *non-degenerate* if $\delta_0 x_0 + \delta_1 x_1 + \ldots + \delta_n x_n$ has no proper vanishing subsum (i.e. if $\sum_{i \in I} \delta_i x_i \neq 0$ for each proper non-empty subset I of $\{0, 1, \ldots, n\}$), and *degenerate* otherwise. Evertse and I proved [4] that the maximal number of pairwise non-proportional non-degenerate solutions of (2) is at most $C = C(n, U)$, where C is a number depending only on n and U. In our Theorem 1 below, this number $C(n, U)$ will be used for $n = 3$.

Theorem 1. *Let $m \geq 3$ be an integer. Then for all but at most*

$$\prod_{i=3}^{m} \binom{i+1}{4} \{6^2 \cdot C(3, U)\}^{m-2} \tag{3}$$

U-equivalence classes of ordered subsets $A = \{\alpha_1, \ldots, \alpha_m\}$ of R, one of the following cases holds:

a) $\mathcal{G}(A)$ is connected and at least one of $\overline{\mathcal{G}(A)}$ and $\overline{\mathcal{G}(A)}^{\square}$ is not connected;

b) $\mathcal{G}(A)$ has two connected components, $\mathcal{G}_1$ and $\mathcal{G}_2$ say, such that $\overline{\mathcal{G}}_1$ is not connected and[2] $|\mathcal{G}_2| = 1$.

Furthermore, if $m = 4$,

c) $\mathcal{G}(A)$ has two connected components of order 2 and $\overline{\mathcal{G}(A)}^{\square}$ is not connected.

This is another version of Theorem 1 of Part I [8] where the same assertion was proved with $\overline{\mathcal{G}(A)}^{\circ}$ instead of $\overline{\mathcal{G}(A)}^{\square}$ and with the bound

$$\{(m+1)! C(m-1, U)\}^{\binom{m}{2}}$$

in place of (3). As was mentioned above, this improvement in terms of m yields much better quantitative results in some applications (cf. [9], [10]).

The proof in [4] does not make it possible to compute C explicitly. We may, however, assume that $C(n, U)$ is a monotonic function of the parameter n. We remark that if we consider in Theorem 1 everywhere $\overline{\mathcal{G}(A)}^{\triangle}$ instead of $\overline{\mathcal{G}(A)}^{\square}$, we can replace in (3) $C(3, U)$ by $C(2, U)$. For $C(2, U)$, an explicit expression was given in [3] (see also Theorem 6 in [8]).

We specialize now our result to the number field case. Next let in particular K be an algebraic number field of degree d over $\mathbb{Q}$, S a finite set

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[2] $|\mathcal{G}|$ denotes the order (number of vertices) of a graph $\mathcal{G}$. Moreover, $|A|$ will denote the cardinality of a finite set A.

of places of K containing all infinite places, s the cardinality of S, $R = \mathcal{O}_S$, the ring of S-integers of K, and $U = \mathcal{O}_S^*$, the group of S-units in K. It has been recently proved by Schlickewei [11] that in this case

$$C(n, \mathcal{O}_S^*) = (4sd!)^{2^{36nd!} \cdot s^6}$$

is an upper bound for the maximal number of pairwise non-proportional, non-degenerate solutions of (2). Very recently, this has been improved in terms of d (see [5]) to

$$C(n, \mathcal{O}_S^*) = (8sd!)^{2^{38nd} \cdot (ns)^6}\,.$$

The factor $d!$ can be replaced by d in both bounds if $K/\mathbb{Q}$ is normal. The proofs of these bounds involve a recent quantitative Subspace Theorem over number fields; see [12]. Defining N, A and $\mathcal{G}(A)$ as in the general case and using the latter value of $C(n, \mathcal{O}_S^*)$ with $n = 3$, we get immediately from Theorem 1 the following

Theorem 2. *Let K, S, d and s be as above, and let $m \geq 3$ be an integer. Then for all but at most*

$$\prod_{i=3}^{m} \binom{i+1}{4} \left\{ (8sd!)^{(m-2)s^6 \cdot 2^{110d}} \right\} \tag{4}$$

$\mathcal{O}_S^*$-equivalence classes of ordered subsets $A = \{\alpha_1, \ldots, \alpha_m\}$ of $\mathcal{O}_S$, one of the cases (a), (b) or (c) (listed in Theorem 1) holds. Further, if $K/\mathbb{Q}$ is normal, $d!$ can be replaced by d in (4).

This theorem will also be used to irreducible polynomials [10] and to pairs of polynomials with given resultant or given semi-resultant [9].

3. Proofs

We keep the notation of Section 2 introduced in the general (finitely generated) case.

Lemma 1. *The maximal number of pairwise non-proportional, non-degenerate solutions of (2) is at most $C = C(n, U)$, where C is a number depending only on n and U.*

Proof. This is Theorem 1 of [4]. ■

Lemma 2. *Let $m \geq 3$ be an integer. There are at most*

$$\prod_{i=3}^{m} \binom{i+1}{4} \{6^2 \cdot C(3,U)\}^{m-2} \tag{5}$$

U-equivalence classes of ordered subsets $A = \{\alpha_1, \ldots, \alpha_m\}$ of R for which both $\overline{\mathcal{G}(A)}$ and $\overline{\mathcal{G}(A)}^{\square}$ are connected.

This is another version of Theorem 5 of [8] which was established with $\overline{\mathcal{G}(A)}^{\circ}$ instead of $\overline{\mathcal{G}(A)}^{\square}$ and with the bound

$$\{(m+1)!C(m-1,U)\}^{\binom{m}{2}}$$

in place of the bound occurring in (5). Following the proof below, one can show that in the above bound of Theorem 5 of [8] the exponent $\binom{m}{2}$ can be replaced by $m-2$.

Proof of Lemma 2. We shall proceed by induction on m. First we prove the assertion for $m = 3$ and $m = 4$ in the case when $\overline{\mathcal{G}(A)}$ has a cycle of length m with property (1). Then $\overline{\mathcal{G}(A)}$ and $\overline{\mathcal{G}(A)}^{\square}$ are obviously connected. Let $m = 3$ or 4, and let $A = \{\alpha_1, \ldots, \alpha_m\}$ be a subset of R for which $\overline{\mathcal{G}(A)}$ has a cycle of length m, say $[\alpha_{i_1}, \alpha_{i_2}], \ldots, [\alpha_{i_{m-1}}, \alpha_{i_m}], [\alpha_{i_m}, \alpha_{i_1}]$, with property (1), where $\{i_1, \ldots, i_m\}$ is a permutation of $\{1, \ldots, m\}$.
Then we have

$$(\alpha_{i_1} - \alpha_{i_2}) + \ldots + (\alpha_{i_{m-1}} - \alpha_{i_m}) + (\alpha_{i_m} - \alpha_{i_1}) = 0. \tag{6}$$

Further, for each edge $[\alpha_p, \alpha_q]$ in the cycle under consideration, we have $\alpha_p - \alpha_q = \delta_{pq} \cdot x_{pq}$ with some $x_{pq} \in U$. Now (6) and (1) imply that $(x_{i_1 i_2}, \ldots, x_{i_{m-1} i_m}, x_{i_m i_1})$ is a non-degenerate solution of the unit equation

$$\delta_{i_1 i_2} x_{i_1 i_2} + \ldots + \delta_{i_{m-1} i_m} x_{i_{m-1} i_m} + \delta_{i_m i_1} x_{i_m i_1} = 0.$$

In view of Lemma 1 we get that

$$\begin{cases} \alpha_p - \alpha_q = \varepsilon \gamma_{pq} & \text{for each } \alpha_p, \alpha_q \text{ with } [\alpha_p, \alpha_q] \text{ belonging} \\ & \text{to the cycle considered above,} \end{cases} \tag{7}$$

where $\varepsilon \in U$, and where the m tuple (γ_{pq}) belongs to a set of m tuples of cardinality at most $C(m-1,U)$ (which set is independent of A). It follows from the connectedness of $\overline{\mathcal{G}(A)}$ that, for every α_p, α_q with $1 \leq p,\ q \leq m$

$(p \neq q)$, there is a path of length at most m from α_p to α_q. Thus (7) implies that

$$\alpha_p - \alpha_q = \varepsilon \rho_{pq} \quad \text{for each } p, q \text{ with } 1 \leq p, \quad q \leq m \quad (p \neq q)$$

where the $\binom{m}{2}$ tuple $(\rho_{pq})'$ belongs to a set of $\binom{m}{2}$ tuples of cardinality at most $C(m-1, U)$ (which set is independent of A). Putting now $A^* = \{0, \rho'_{21}, \ldots, \rho_{m1}\}$, we get that $A = \varepsilon A^* + \alpha_1$, where A^* belongs to a finite subset of R^m of cardinality at most $C(m-1, U)$. The number of distinct cycles under consideration is at most $(m-1)!$. Thus, for $m = 3$ and 4, the number of U-equivalence classes of ordered subsets A of R for which $\overline{\mathcal{G}(A)}$ is connected and $\overline{\mathcal{G}(A)}^{\square}$ has a cycle of length m with property (1) is at most

$$(m-1)!C(m-1, U), \tag{8}$$

which implies the assertion of the lemma.

Suppose now that $m > 3$, and that the assertion has been proved for each integer m' with $3 \leq m' < m$. Consider an arbitrary ordered subset $A = \{\alpha_1, \ldots, \alpha_m\}$ of R for which both $\overline{\mathcal{G}(A)}$ and $\overline{\mathcal{G}(A)}^{\square}$ are connected. If $m = 4$, it is enough to consider the case when $\overline{\mathcal{G}(A)}$ has not a cycle of length 4 with property (1). From the connectedness of $\overline{\mathcal{G}(A)}$ and $\overline{\mathcal{G}(A)}^{\square}$ it follows that there is a subset A' of A with the following properties: A' has cardinality $m' = m-2$ or $m' = m-1$, $\overline{\mathcal{G}(A)}$ has a subgraph $\mathcal{H}(A')$ with vertex set A' such that $\mathcal{H}(A')$ and $\mathcal{H}(A')^{\square}$ are connected, and for each cycle $\mathcal{H}_0(A_0)$ in $\overline{\mathcal{G}(A)}$ with length 3 or 4 and with property (1) for which the vertex set A_0 of $\mathcal{H}_0(A_0)$ is not contained in A' and $\mathcal{H}_0(A_0)$ and $\mathcal{H}(A')$ have a common edge, we have $A' \cup A_0 = A$. Then, by our induction hypothesis,

$$A' = \varepsilon A'^* + \beta \tag{9}$$

with some $\varepsilon \in U$, $\beta \in R$ and some A'^* belonging to a finite subset of $R^{m'}$ of cardinality at most

$$\prod_{i=3}^{m'} \binom{i+1}{4} \{6^2 \cdot C(3, U)\}^{m'-2}. \tag{10}$$

$\overline{\mathcal{G}(A)}$ and $\overline{\mathcal{G}(A)}^{\square}$ being connected, there is a cycle $\mathcal{H}_0(A_0)$ in $\overline{\mathcal{G}(A)}$ with vertex set A_0, with length $m'' = 3$ or 4 and with property (1) such that

$A' \cup A_0 = A$ and that $\mathcal{H}(A')$ and $\mathcal{H}_0(A_0)$ have a common edge. Then, as we showed above,

$$A_0 = \rho A_0^* + \sigma \tag{11}$$

with some $\rho \in U$, $\sigma \in R$ and with an A_0^* belonging to a finite subset of $R^{m''}$ of cardinality at most

$$(m''-1)!C(m''-1, U) \leq 6 \cdot C(3, U).$$

Suppose that $A_0 = \{\alpha_{i_1}, \ldots, \alpha_{i_{m''}}\}$ and that $[\alpha_{i_1}, \alpha_{i_2}]$ is a common edge of $\mathcal{H}(A')$ and $\mathcal{H}_0(A_0)$. Then it follows from (9) and (11) that

$$\alpha_{i_j} = \varepsilon\alpha_{i_j}^* + \beta = \rho\kappa_{i_j}^* + \sigma \quad for \quad j = 1, 2 \tag{12}$$

and

$$\alpha_{i_j} = \rho\kappa_{i_j}^* + \sigma \quad for \quad j = 3, \ldots, m'' \tag{13}$$

where $\alpha_{i_j}^*$ and $\kappa_{i_j}^*$ are the corresponding coordinates of A'^* and A_0^*, respectively. Now (12) and (13) imply that for fixed A'^*,

$$\alpha_{i_j} = \varepsilon\alpha_{i_j}' + \beta \quad for \quad j = 3, \ldots, m'',$$

where the tuple (α_{i_j}') belongs to a finite subset of $R^{m''-2}$ of cardinality at most $6 \cdot C(3, U)$. Consequently,

$$\alpha_i = \varepsilon\alpha_i^0 + \beta \quad for \quad i = 1, \ldots, m,$$

where $(\alpha_1^0, \ldots, \alpha_m^0)$ belongs to a finite subset of R^m of cardinality at most

$$6^{2(m-3)+1} \cdot \prod_{i=3}^{m-1} \binom{i+1}{4} \{6^2 \cdot C(3, U)\}^{m-2}. \tag{14}$$

But the number of choices for A' is at most $\binom{m}{m'}$. If $m' = m-2$ then for fixed A', the number of choices for A_0 does not exceed the number of edges of A', that is $\binom{m'}{2}$. In this case, the number of choices for A' and A_0 altogether is at most $\binom{m}{m'} \cdot \binom{m'}{2} \leq 6\binom{m}{4}$. For $m' = m-1$, the number of choices for A_0 is at most $\binom{m'}{3}$ if $m' \geq 5$, and then the number of choices for A' and A_0 altogether is at most $\binom{m}{m'} \cdot \binom{m'}{3} \leq 4\binom{m}{4}$. It is easy to see that this bound is also valid for $m' = 3$ and 4. Consequently, for every ordered subset A of R under consideration which has cardinality m can be written in the from $A = \Theta A^* + \omega$, where $\Theta \in U$, $\omega \in R$ and where A^* belongs to a finite subset of R^m of cardinality at most

$$\prod_{i=3}^{m} \binom{i+1}{4} \{6^2 \cdot C(3,U)\}^{m-2}$$

which completes the proof of the lemma. ∎

Proof of Theorem 1. We shall use some arguments from the proof of Theorem 1 of [8]. In fact, it would be enough to point out the differences in that proof and the present one. However, for convenience of the reader, we give here a complete proof for our theorem.

Let $A = \{\alpha_1, \dots, \alpha_m\}$ be an arbitrary but fixed ordered subset of R such that at least one of $\overline{\mathcal{G}(A)}$ and $\overline{\mathcal{G}(A)}^{\square}$ is not connected. By Lemma 2, it suffices to prove that for $\mathcal{G}(A)$, (a), (b) or (c) holds. We denote by l the number of connected components of $\mathcal{G}(A)$. It is easy to see that, for $l \geq 3$, both $\overline{\mathcal{G}(A)}$ and $\overline{\mathcal{G}(A)}^{\triangle}$ (and hence also $\overline{\mathcal{G}(A)}^{\square}$) would be connected. Therefore we have $l \leq 2$.

For $l = 1$, case (a) holds. In what follows, we consider the case $l = 2$. Then $\overline{\mathcal{G}(A)}$ is connected and hence, by assumption, $\overline{\mathcal{G}(A)}^{\square}$ cannot be connected. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the connected components of $\mathcal{G}(A)$ with $|\mathcal{G}_1| \geq |\mathcal{G}_2|$. In the case $|\mathcal{G}_2| = 1$, $\overline{\mathcal{G}_1}$ cannot be connected since otherwise $\mathcal{G}(A)^{\triangle}$ (and hence also $\overline{\mathcal{G}(A)}^{\square}$) would be connected. Thus, in this case we get case (b). It remained to consider the case when $|\mathcal{G}_2| \geq 2$.

We repeat now the corresponding part of the proof of Theorem 1 of [8]. First suppose that $|\mathcal{G}(A)| \geq 5$. Let A_1 and A_2 be arbitrary subsets of the vertex sets of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, such that $2 \leq |A_2| \leq |A_1|$ and $|A_1| \geq 3$. Then $\overline{\mathcal{G}(A_1 \cup A_2)}$ is obviously connected. Further, putting $t = |A_1 \cup A_2|$, we have $t \geq 5$. We show now by induction on t that $\overline{\mathcal{G}(A_1 \cup A_2)}^{\square}$ is also connected. We prove first this assertion for $t = 5$. Let $A_1 = \{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}\}$ and $A_2 = \{\alpha_{i_4}, \alpha_{i_5}\}$. The sum

$$(\alpha_{i_4} - \alpha_{i_1}) + (\alpha_{i_1} - \alpha_{i_5}) + (\alpha_{i_5} - \alpha_{i_2}) + (\alpha_{i_2} - \alpha_{i_4}) \tag{15}$$

can have a proper vanishing subsum (with summands $(\alpha_{i_4} - \alpha_{i_1}), (\alpha_{i_1} - \alpha_{i_5})$, $(\alpha_{i_5} - \alpha_{i_2})$ and $(\alpha_{i_2} - \alpha_{i_4})$) only if

$$0 = (\alpha_{i_4} - \alpha_{i_1}) + (\alpha_{i_5} - \alpha_{i_2}) = (\alpha_{i_1} - \alpha_{i_5}) + (\alpha_{i_2} - \alpha_{i_4}). \tag{16}$$

Similarly, the only possible proper vanishing subsum of

$$(\alpha_{i_4} - \alpha_{i_1}) + (\alpha_{i_1} - \alpha_{i_5}) + (\alpha_{i_5} - \alpha_{i_3}) + (\alpha_{i_3} - \alpha_{i_4}) \tag{17}$$

(with summands $(\alpha_{i_4} - \alpha_{i_1})$, $(\alpha_{i_1} - \alpha_{i_5})$, $(\alpha_{i_5} - \alpha_{i_3})$ and $(\alpha_{i_3} - \alpha_{i_4})$) are

$$0 = (\alpha_{i_4} - \alpha_{i_1}) + (\alpha_{i_5} - \alpha_{i_3}) = (\alpha_{i_1} - \alpha_{i_5}) + (\alpha_{i_3} - \alpha_{i_4}). \tag{18}$$

But $\alpha_{i_2} \neq \alpha_{i_3}$. Hence (16) and (18) cannot hold simultaneously. Hence, at least one of the quadruplets $\alpha_{i_4}, \alpha_{i_1}, \alpha_{i_5}, \alpha_{i_2}$, and $\alpha_{i_4}, \alpha_{i_1}, \alpha_{i_5}, \alpha_{i_3}$ is a cycle in $\overline{\mathcal{G}(A_1 \cup A_2)}$ with property (1). We may assume that $\alpha_{i_4}, \alpha_{i_1}, \alpha_{i_5}, \alpha_{i_2}$ is a cycle with property (1). One can prove in a similar way that at least one of the quadruplets $\alpha_{i_4}, \alpha_{i_3}, \alpha_{i_5}, \alpha_{i_1}$ and $\alpha_{i_4}, \alpha_{i_3}, \alpha_{i_5}, \alpha_{i_2}$ is also a cycle in $\overline{\mathcal{G}(A_1 \cup A_2)}$ with property (1). Thus it follows that $\overline{\mathcal{G}(A_1 \cup A_2)}^{\square}$ is connected. Assume now that $t > 5$ and that the assertion has been proved for each integer t' with $5 \geq t' < t$. Let again $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3} \in A_1$ and $\alpha_{i_4}, \alpha_{i_5} \in A_2$. For $|A_1| > |A_2|$, let $A_1' = A_1 \setminus \{\alpha_{i_1}\}$ and $A_2' = A_2$ and, for $|A_1| = |A_2|$, let $A_1' = A_2$ and $A_2' = A_1 \setminus \{\alpha_{i_1}\}$. Then, by the inductive hypothesis, both $\overline{\mathcal{G}(A_1' \cup A_2')}^{\square}$ and $\overline{\mathcal{G}(\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4}, \alpha_{i_5}\})}^{\square}$ are connected. It is easy to show that then $\overline{\mathcal{G}(A_1 \cup A_2)}^{\square}$ is also connected. Finally, it follows that $\overline{\mathcal{G}(A)}^{\square}$ is connected which contradicts our assumption.

It remained the case when $l = 2$, $|\mathcal{G}_1| = |\mathcal{G}_2| = 2$ and $\overline{\mathcal{G}(A)}^{\square}$ is not connected. This is, however, just case (c). ■

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