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Ramsey Numbers for Graph Sets with Fixed Numbers of Edges

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The classical Ramsey number $r = r(G, H)$ is the smallest r, such that every 2–coloring of the edges of the complete graph K_r contains a graph G with all edges of color 1, or a graph H with all edges of color 2. If $G = H$ then $r(G, H) = r(G)$ is used.

It is frustrating that $r(K_3) = 6$ can be proved even by non-mathematicians, that $r(K_4) = 18$ is not too hard to prove, and that already $r(K_5)$ is still unknown. For complete graphs, the only further known values are $r(K_3, K_n) = 9, 14, 18, 23, 28, 36$ for $n = 4, 5, \ldots, 9$. Recently some progress was made on the estimations of $r(K_4, K_5)$ and $r(K_5)$ (see [14]):

$$25 \leq r(K_4, K_5) \leq 27 \qquad \text{and} \qquad 43 \leq r(K_5) \leq 52.$$

For small graphs with $|G| \leq 4$ and $|H| \leq 4$ all Ramsey numbers are given in [3]. In [4] all but five values for $|G| = 4$ and $|H| = 5$ are listed, and in [1, 6, 12] the missing values up to $r(K_4, K_5)$ are determined. In [7] all numbers $r(K_3, H)$ are given for $|H| = 6$. All diagonal Ramsey numbers $r(G)$ are listed in [2] if G has at most 6 edges, and in [11] if G has 7 edges. A table with all but seven values $r(G, H)$ with $|G| = |H| = 5$ is presented in [13]. There are 23 such graphs without isolated vertices. The largest Ramsey number in this table is $r(K_5 - e) = 22$ (see [5]). The only known number with $|G| = 6$ and more than 7 edges seems to be $r(K_{3,3}) = 18$ (see [10]).

If one hopes to get some insight in $r(K_n)$ by considering $r(G, H)$ for all smaller graphs G and H then very soon there are too many graphs on the way to $r(K_n)$. Therefore it was proposed already in [1] to discuss $r = r_{m,n}(s,t)$, that means, to determine the smallest r such that every 2–coloring of the edges of K_r contains any graph with m vertices and s edges of color 1, or any graph with n vertices and t edges of color 2; $1 \leq s \leq \binom{m}{2}$ and $1 \leq t \leq \binom{n}{2}$. Thus it is asked for an $\binom{m}{2}$ by $\binom{n}{2}$ rectangular array of Ramsey numbers for graph sets with fixed numbers of edges. Note that the last two rows and columns contain $r(K_m - e, K_n - e), r(K_m - e, K_n), r(K_m, K_n - e)$ and $r(K_m, K_n)$. As examples see Table 1 for $r_{3,7}(s,t)$ ([8]), and Table 2 for $r_{4,5}(s,t)$ ([1]).

$s \setminus t$	...	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1		7	7	7	7	7	7	7	7	7	7	7	7	7	7
2		7	7	7	7	7	7	7	7	7	7	7	9	11	13
3		7	7	9	9	11	11	11	11	13	13	14	17	21	23

Table 1. $r_{3,7}(s,t)$.

$s \setminus t$	1	2	3	4	5	6	7	8	9	10
1	5	5	5	5	5	5	5	5	5	5
2	5	5	5	5	5	5	5	5	5	6
3	5	5	5	5	5	5	5	6	7	9
4	5	5	5	5	5	6	7	8	11	13
5	5	5	5	5	7	7	10	11	13	16
6	5	5	7	7	10	10	13	14	19	

Table 2. $r_{4,5}(s,t)$.

Here first general results for $r_{m,n}(s,t)$ are given in the case $m = n$, that is, for $r_{n,n}(s,t) = r_n(s,t)$. Since $r_n(s,t) = r_n(t,s)$, the values below the diagonal are known by symmetry. It is easy to see that

$$r_n(s,t) = n \qquad \text{for} \qquad s + t \leq 1 + \binom{n}{2}, \tag{1}$$

so that only values below the secondary diagonal and not below the diagonal remain to be determined.

Theorem 1. *For $n \geq 3$ and $2 \leq s \leq 1 + \frac{1}{2}\binom{n}{2}$ holds*

$$r_n\left(s, \binom{n}{2} - s + 2\right) = \begin{cases} n+2 & \text{if } (n-1)|2(s-1), \\ n+1 & \text{otherwise.} \end{cases}$$

This means, on the first parallel line to the secondary diagonal every $(n-1)$–st Ramsey number, and for odd n the central number of every interval of two numbers $n+2$ also, has the value $n+2$. The following theorem determines the values on the second parallel line to the secondary diagonal.

Theorem 2. *For $n \geq 3$ and $3 \leq s \leq \frac{1}{2}(3 + \binom{n}{2})$ holds*

$$r_n\left(s, \binom{n}{2} - s + 3\right) = \begin{cases} n+4 & \text{if } s = \frac{n+1}{2}, \quad n \equiv 1 \pmod 2, \\ n+3 & \text{if } (n-2)|2(s-2), \\ n+2 & \text{otherwise.} \end{cases}$$

As a consequence of Theorems 1 and 2 the first nontrivial Ramsey number on the main diagonal is as follows:

Corollary. *For $n \geq 3$ holds*

$$r_n\left(1 + \left\lceil \frac{1}{2}\binom{n}{2} \right\rceil, 1 + \left\lceil \frac{1}{2}\binom{n}{2} \right\rceil\right) = \begin{cases} n+2 & \text{if } n \equiv 0 \pmod 2, \\ n+1 & \text{if } n \equiv 1 \pmod 4, \\ n+3 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

For the proofs of Theorems 1 and 2 the following lemma is useful.

Lemma. *For fixed d, $1 \le d \le p-3$, no p-graph does exist such that every $(p-1)$-subgraph is regular of degree d.*

Proof. Let G be a p-graph. No vertex v of G has degree 0 or $p-1$ since the deletion of any other vertex would leave a $(p-1)$-subgraph of G where v has degree 0 or $p-2$, respectively. Thus in G at least one vertex w_1 exists which is, and one vertex $w_2 \ne w_1$ which is not connected to any vertex v in G. The deletion of w_1 or w_2, however, implies different degrees of v in $(p-1)$-subgraphs of G. ∎

Proof of Theorem 1. Every n-subgraph of K_{r-1} with $r = r_n(s, \binom{n}{2} - s + 2)$ has exactly $s-1$ edges of color 1 if K_{r-1} is 2-colored such that it does not contain an n-subgraph with s edges of color 1 or $\binom{n}{2} - s + 2$ edges of color 2. Then every $(n+1)$-subgraph of K_{r-1} contains $(n+1)(s-1)/(n-1)$ edges of color 1, and all vertices are of degree

$$d = \frac{(n+1)(s-1)}{n-1} - (s-1) = \frac{2(s-1)}{n-1} \le \frac{n}{2}.$$

Thus $d \le (n+2) - 3$, and the Lemma implies $r - 1 < n + 2$, that is, $r \le n+2$ if d is an integer, and $r \le n+1$ otherwise.

The lower bound, $r \ge n+1$, follows immediately since $r_n(s,t) = n$ if and only if $s + t \le 1 + \binom{n}{2}$. For $r \ge n+2$ the edges of color 1 determine a regular $(n+1)$-graph of degree $d = 2(s-1)/(n-1)$ as follows: The vertices of a convex $(n+1)$-gon are connected $\lfloor d/2 \rfloor$-times by all $n+1$ diagonals of the same distance ($\le \frac{n}{2}$), and for d odd, which only occurs for $n+1$ even, in addition the main diagonals are used. ∎

Proof of Theorem 2. ($\le$) Every n-subgraph of K_{r-1} with $r = r_n(s, \binom{n}{2} - s + 3)$ has $s-2$ or $s-1$ edges of color 1 if K_{r-1} is 2-colored such that it does not contain an n-subgraph with s edges of color 1 or $\binom{n}{2} - s + 3$ edges of color 2. For any $(n+1)$-subgraph G_{n+1} of K_{r-1} let a be the number of n-subgraphs with $s-2$ edges of color 1, $0 \le a \le n+1$. Then the number C_1 of edges of color 1 in G_{n+1} is

$$C_1 = \frac{a(s-2) + (n+1-a)(s-1)}{n-1} = s - 1 + \frac{2(s-1) - a}{n-1},$$

and there are a vertices in G_{n+1} of color-1-degree $d_G = C_1 - (s-2)$, and $n+1-a$ vertices of color-1-degree $d_G - 1$. Since C_1 is an integer, every G_{n+1} has the same value for C_1 if $a = 3, 4, \dots,$ or $n-2$, and for $n \ge 4$

there are two values for C_1 possible if $a = 0$ or $a = n - 1$, if $a = 1$ or $a = n$, and if $a = 2$ or $a = n + 1$, whereas for $n = 3$ there are the three possibilities $a = 0$, $a = n - 1 = 2$, and $a = n + 1 = 2(n - 1) = 4$.

If every G_{n+1} has the same value for C_1, that is, the same number a of n-subgraphs with $s - 2$ edges of color 1 for $2 \leq a \leq n$, then the number C_2 of edges of color 1 in any $(n + 2)$-subgraph G_{n+2} of K_{r-1} is $C_2 = (n + 2)C_1/n$, and all the vertices of G_{n+2} have the same color-1-degree $D = C_2 - C_1 = 2C_1/n$. Then, however, D also is equal to the larger color-1-degree in G_{n+1}, that is $D = C_1 - (s - 2)$, and altogether follows $D = 2(s - 2)/(n - 2) \leq (n + 1)/2$. Then $1 \leq D \leq (n + 3) - 3$, and the Lemma implies $r - 1 < n + 3$, that is, $r \leq n + 3$ if D is an integer, and $r \leq n + 2$ otherwise.

It remain the three cases where at least one $(n+1)$-subgraph H of K_{r-1} exists for which the number a of n-subgraphs with $s - 2$ edges of color 1 is $a = 0, a = 1$, or $a = n + 1$.

$\underline{a = 0}$: Every n-subgraph of H has $s - 1$ edges of color 1, and the color-1-degree for all $n + 1$ vertices of H is $d_H - 1 = d - 1 = 2(s - 1)/(n - 1)$ with $1 \leq d - 1 < n$. All other subgraphs G_{n+1} of K_{r-1} have color-1-degree $d - 1$, or $d - 2$.

Every vertex x of H is connected by an edge of color 2 to any vertex w outside H, since otherwise the deletion of a vertex y of H, with (x, y) of color 2, would yield a G_{n+1} with color-1-degree d for x. Thus the color-1-degree of w for that G_{n+1}, consisting of w together with n vertices of H, equal 0, and $0 \geq d - 2$ yields $d = 2$ which corresponds to $s = (n + 1)/2$, $n \equiv 1 \pmod 2$.

Now H contains $(n + 1)/2$ disjoint edges of color 1. Since $s \geq 3$, the deletion of one vertex of each of three disjoint edges together with any three vertices outside H would determine a subgraph G_{n+1} with three vertices of color-1-degree $0 = d - 2$ instead of two. This contradiction proves $r - 1 \leq n + 3$.

$\underline{a = 1}$: There is one vertex in H with color-1-degree $d_H = d = 1 + (2(s - 1) - 1)/(n - 1)$, $d \geq 2$, and n vertices have color-1-degree $d - 1$. The color-1-degrees of all other G_{n+1} are either n-times $d - 1$ and once d, or n-times $d - 1$ and once $d - 2$.

Assume, a vertex w of K_{r-1} exists outside H. Let x, y, z be vertices of H with x of color-1-degree d, (x, y) of color 2, and (y, z) of color 1. Then (w, x) is of color 2 (delete y), (w, y) is of color 2 (delete x, and w so as y

would be of color-1-degree d), and (w, z) is of color 1 (delete y). Now the deletion of z leaves a G_{n+1} with vertices w and z both of color-1-degree $d-2$. This is impossible, and $r-1 \leq n+1$ is proved.

$\underline{a = n+1:}$ Every vertex of H has color-1-degree $d_H = d = 2(s-2)/(n-1)$ with $1 \leq d < n-1$. Every other G_{n+1} of K_{r-1} has color-1-degrees d or $d+1$.

Assume, a vertex w of K_{r-1} exists outside H. For any vertex x of H the edge (w, x) is of color 1 (delete a vertex y of H with (y, x) of color 1, existence follows from $d \geq 1$). Then w has color-1-degree n in every G_{n+1} with vertices w and n vertices of H, however, $n \leq d+1$ contradicts $d < n-1$. Thus w does not exist, and $r-1 \leq n+1$.

($\geq$) A K_{n+3} with $(n+3)/2$ disjoint edges of color 1 and all other edges of color 2 proves $r \geq n+4$ in case n odd and $s = (n+1)/2$.

For $r \geq n+3$ a coloring of K_{n+2} is used where, as in the proof of Theorem 1, the edges of color 1 determine a regular $(n+2)$-graph of degree $2(s-2)/(n-2)$.

For $r \geq n+2$ consider $x = \lceil 2(s-2)/(n-1) \rceil$ and $y = (n-1)x - 2(s-2)$.

If $n \equiv 0 \pmod 2$, consider a partition of K_{n+1} into Hamiltonian cycles. If $x \equiv 1 \pmod 2$, that is, $y \equiv 1 \pmod 2$ and $n+1-y \equiv 0 \pmod 2$, then for $(x-1)/2$ Hamiltonian cycles, so as for $(n+1-y)/2$ disjoint edges of another Hamiltonian cycle, color 1 is used. If $x \equiv 0 \pmod 2$, that is, $y \equiv 0 \pmod 2$, then $(x-2)/2$ Hamiltonian cycles so as from another Hamiltonian cycle for all edges excluded $y/2$ disjoint of them, color 1 is used.

If $n \equiv 1 \pmod 2$, that is $y \equiv 0 \pmod 2$ and $n+1-y \equiv 0 \pmod 2$, then K_{n+1} is partitioned into matchings, and $x-1$ of them, so as $(n+1-y)/2$ edges of another matching, are of color 1.

Every coloring of K_{n+1} has $s-2+x$ edges of color 1, the color-1-degrees of the vertices are x or $x-1$, and thus every n-subgraph has $s-2$ or $s-1$ edges of color 1. ■

The proof of Theorem 2 is complete. — Perhaps the third parallel line of the secondary diagonal can be determined in a similar way.

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