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On Extremal Problems Concerning Weights of Edges of Graphs

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Our terminology and notation will be standard as indicated. A good reference for undefined terms is Harary [4]. For two disjoint graphs G_1 and G_2 $G_1 \oplus G_2$ and $G_1 \cup G_2$ will denote their join (Zykov sum) and their union, respectively.

The weight $w(e)$ of an edge $e = uv$ of a graph G is defined to be the sum of degrees of the vertices u, v. This concept of the weight of an edge was introduced by Kotzig [7] who proved the following beautiful result: Every planar 3-connected graph contains an edge of the weight not exceeding 13. This result was further developed in various directions. Grünbaum [2], Jucovič [6] and recently Borodin [1] have studied the inequalities for the numbers of edges having weights not exceeded 13 in planar 3-connected graphs. Ivančo [5] has found an analogue of Kotzig's result for graphs with the minimum degree at least 3 and embedded on orientable 2-manifolds. The analogue of Kotzig's result for triangulations of orientable 2-manifolds can be found in Zaks [8] and for periodical edge to edge tiling of the plane in Grünbaum and Shephard [3].

At the Fourth Czechoslovak Symposium on Combinatorics held in Prachatice 1990 professor Erdős asked the question: What is the minimum weight of an edge e of a graph G having n vertices and m edges?

Throughout the paper let n, m be integers such that $n \geq 2$, $0 \leq m \leq \binom{n}{2}$ and let $\mathcal{G}(n, m)$ be the family of all graphs having n vertices and m edges.

Motivated by the papers mentioned and Erdős's question we consider the problem: What is the necessary weight of an edge of a graph $G = (V, E)$ from $\mathcal{G}(n, m)$?

If we denote by

$$w(n,m) = \min_{G \in \mathcal{G}(n,m)} \left\{ \max_{e \in E(G)} \{w(e)\} \right\} \tag{1}$$

and

$$W(n,m) = \max_{G \in \mathcal{G}(n,m)} \left\{ \min_{e \in E(G)} \{w(e)\} \right\} \tag{2}$$

then easy observations provide

Lemma 1.1. *Every graph $G \in \mathcal{G}(n, m)$ contains edges h_1 and h_2 such that*

$$w(n,m) \leq w(h_1) \quad \text{and} \quad w(h_2) \leq W(n,m). \ \blacksquare$$

The answer to the problem of Erdős is now in finding out the value $W(n, m)$. The paper is devoted to study both above defined numbers and two other numbers derived.

Note that in the sequel $\lceil x \rceil$ and $\lfloor x \rfloor$ will denote the upper and the lower integer part of x.

2. Sum of the weights of edges

For $G \in \mathcal{G}(n, m)$ let $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$ be the vertex set and the edge set of the graph G, respectively.

Let

$$F(G) = \sum_{i=1}^{m} w(e_i). \tag{3}$$

In this section we determine the following two values

$$f(n,m) = \min \{F(G) | \quad G \in \mathcal{G}(n,m)\} \tag{4}$$

and

$$F(n,m) = \max\{F(G) | \quad G \in \mathcal{G}(n,m)\}. \tag{5}$$

First some properties of $F(G)$ are considered.

Let us denote $\deg(v_i) = d_i$ for every $i = 1,2,\ldots,n$. Clearly $d_1 + d_2 + \ldots + d_n = 2m$. Since a vertex v_i contributes to the weight of d_i edges we have

$$F(G) = w(e_1) + \ldots + w(e_m) = d_1^2 + \ldots + d_n^2. \tag{6}$$

Lemma 2.1. *Let $G = (V,E) \in \mathcal{G}(n,m)$ and let for $v_i, v_j, v_k \in V$ there is $v_iv_j \in E$ and $v_jv_k \notin E$. Therefore the graph $G^* = (V,E^*)$ such that $E^* = (E - \{v_iv_j\}) \cup \{v_jv_k\}$ it holds $G^* \in \mathcal{G}(n,m)$ and $F(G^*) = F(G) + 2(d_k - d_i) + 2$.*

Proof. Let $d_1^*,\ldots,d_n^*$ be a degree sequence of the graph G^*. Clearly $d_s^* = d_s$ for every $s \neq i,k, d_i^* = d_i - 1$ and $d_k^* = d_k + 1$. By (6) then we have

$$F(G^*) = \sum_{s=1}^{n}(d_s^*)^2 = \sum_{s=1, s\neq i,k}^{n} d_s^2 + (d_i - 1)^2 + (d_k+1)^2 = F(G) + 2(d_k - d_i) + 2.$$

The rest property of G^* is obvious. ■

Lemma 2.1 immediately provides

Lemma 2.2. *Let for $G \in \mathcal{G}(n,m)$ $d_1 \geq d_2 \geq \ldots \geq d_n$ and $F(G) = F(n,m)$. Then there is*

$$d_1 = \max\{i \,|\, d_i > 0\} - 1.$$

Lemma 2.3. *(i) Let $G_1, G_2 \in \mathcal{G}(n,m)$. If $F(G_1) \leq F(G_2)$, then $F(K_1 \oplus G_1) \leq F(K_1 \oplus G_2)$. (ii) If $F(K_1 \oplus G) = F(n,m)$ then $G \in \mathcal{G}(n-1, m-n+1)$‘ and $F(G) = F(n-1, m-n+1)$.*

Lemma 2.4. *Let $G \in \mathcal{G}(n,m)$ and $F(G) = F(n,m)$ then*

$$F(\overline{G}) = F\left(n, \binom{n}{2} - m\right).$$

Proof. Clearly $\overline{G} \in \mathcal{G}(n, \binom{n}{2} - m)$. $F(\overline{G}) = \sum_{i=1}^{n}(n-1-d_i)^2 = n(n-1)^2 - 2(n-1)2m + F(G)$.

Suppose $G_1 \in \mathcal{G}(n, \binom{n}{2} - m)$ and $F(G_1) > F(\overline{G})$. Since $\overline{G}_1 \in \mathcal{G}(n,m)$ and $F(G) = F(n,m)$ we have $F(\overline{G}_1) \leq F(G)$. This means $F(\overline{G}) < F(G_1) =$

$n(n-1)^2 - 4(n-1)m + F(\overline{G}_1) \le n(n-1)^2 - 4(n-1)m + F(G) = F(\overline{G})$, a contradiction. ■

For $n \ge 2$ let $\mathbf{M}_n = \{(x_1, x_2, \dots, x_{n-1}) \mid x_i \in \{0,1\}\}$. For $\mathbf{x} = (x_1, \dots, x_{n-1}) \in \mathbf{M}_n$ we define the graph $G_\mathbf{x} = (V,E)$ as follows: $V = \{v_1, v_2, \dots, v_n\}$ and $v_i v_j \in E$ with $i < j$ if and only if $x_i = 1$. Let $\mathcal{M}_n = \{G_\mathbf{x} \mid \mathbf{x} \in \mathbf{M}_n\}$.

Lemma 2.5. *If $G \in \mathcal{G}(n,m)$ and $F(G) = F(n,m)$ then $G \in \mathcal{M}_n$.*

Proof is by induction on n. For $n = 2$ the graphs $G_{(0)}$ and $G_{(1)}$ satisfy the Lemma. Assume that Lemma is true for an integer $n-1$, let $G \in \mathcal{G}(n,m)$ and $F(G) = F(n,m)$.

If G is connected then, by Lemma 2.2, $G = K_1 \oplus G_1$ and, by Lemma 2.3 (ii), $F(G_1) = F(n-1, m-n+1)$. By the induction hypothesis $G_1 = G_{\mathbf{x}'}$ for some $\mathbf{x}' = (x'_1, \dots, x'_{n-2}) \in \mathbf{M}_{n-1}$. Let $\mathbf{x} = (1, x'_1, x'_2, \dots, x'_{n-2})$ then $\mathbf{x} \in \mathbf{M}_n$ and $G_\mathbf{x} = G$.

If G is disconnected, then $\overline{G}$ is connected and by Lemma 2.4 $F(\overline{G}) = F(n, \binom{n}{2} - m)$. By Lemma 2.2 there is $\overline{G} = K_1 \oplus \overline{G}_1$ for a graph $\overline{G}_1$ such that $F(\overline{G}_1) = F(n-1, \binom{n}{2} - m - n + 1)$. Then $G = K_1 \cup G_1$ with $G_1 \in \mathcal{G}(n-1, m)$ and $F(G_1) = F(n-1, m)$. By the induction hypothesis $G_1 = G_{\mathbf{x}'}$ for some $\mathbf{x}' = (x'_1, \dots, x'_{n-1}) \in \mathbf{M}_{n-1}$. Let $\mathbf{x} = (0, x'_1, x'_2, \dots, x'_{n-2})$, clearly $\mathbf{x} \in \mathbf{M}_n$ and $G = G_\mathbf{x}$. ■

Before starting the next Lemma it is convenient to introduce some notations. The $(n-1)$-tuple $\overline{\mathbf{x}} = (1 - x_1, \dots, 1 - x_{n-1})$ for a $(n-1)$-tuple $\mathbf{x} = (x_1, \dots, x_{n-1})$ will be called to be complement to $\mathbf{x}$ in the sequel. The record of the $(n-1)$-tuple $\mathbf{x} \in \mathbf{M}_n$ we will abbreviate by writing $i(k)$ and $o(k)$ instead of k consecutive units and k consecutive zeros, respectively.

Let $\mathbf{V}_n$ be the subfamily of the family $\mathbf{M}_n$ such that an $(n-1)$-tuple $\mathbf{t}$ belongs to $\mathbf{V}_n$ if $\mathbf{t}$ has one of next 12 forms.

I	$\mathbf{t} = (i(n-1))$	I*	$\mathbf{t} = (o(n-1))$
II	$\mathbf{t} = (i(k), o(s))$	II*	$\mathbf{t} = (o(k), i(s))$
III	$\mathbf{t} = (i(k), o(s), i(1))$	III*	$\mathbf{t} = (o(k), i(s), o(1))$
IV	$\mathbf{t} = (i(k), o(s), i(1), o(j))$	IV*	$\mathbf{t} = (o(k), i(s), o(1), o(j))$
V	$\mathbf{t} = (o(s), i(1))$	V*	$\mathbf{t} = (i(s), o(1))$
VI	$\mathbf{t} = (o(s), i(1), o(j))$	VI*	$\mathbf{t} = (i(s), o(1), i(j))$

(j, k, s are positive integers).

Lemma 2.6. *For every $n \ge 2$ and $0 \le m \le \binom{n}{2}$ there exists $\mathbf{x} \in \mathbf{V}_n$ such that $F(G_\mathbf{x}) = F(n,m)$.*

Proof is by induction on n. For $n = 2$ and 3 it is clear. Suppose Lemma is true for a positive integer n. Consider a graph $G \in \mathcal{G}(n+1, m)$ such that $F(G) = F(n+1, m)$. Because the claim $F(G) = F(n+1, m)$ is equivalent to the claim $F(\overline{G}) = F(n+1, \binom{n+1}{2} - m)$ and $\overline{G}_{\mathbf{x}} = G_{\overline{\mathbf{x}}}$ we can assume that G is connected. By Lemma 2.3 we have $G = K_1 \oplus (G - v)$ and $F(G - v) = F(n, m - n)$. By the induction hypothesis there exists $\mathbf{z} \in \mathbf{V}_n$ such that $G_{\mathbf{z}} \in \mathcal{G}(n, m - n)$ and $F(G_{\mathbf{z}}) = F(n, m - n)$. By Lemma 2.3 $F(K_1 \oplus G_{\mathbf{z}}) = F(G) = F(n+1, m)$.

Let us consider the n-tuple $\mathbf{x} = (1, \mathbf{z})$. Clearly $\mathbf{x} \in \mathbf{M}_{n+1}$, $G_{\mathbf{x}} = K_1 \oplus G_{\mathbf{z}} \in \mathcal{G}(n+1, m)$ and $F(G_{\mathbf{x}}) = F(n+1, m)$. It is easy to see that for $\mathbf{z}$ being of the form I, II, III, IV, V, VI, I*, V* or VI* (respectively) $\mathbf{x} \in \mathbf{V}_{n+1}$. For the rest cases see Table below where for $\mathbf{x}$ considered the n-tuple $\mathbf{y} \in \mathbf{M}_{n+1}$ is determined in such a way that $\mathbf{y}$ has either the properties $G_{\mathbf{y}} \in \mathcal{G}(n+1, m)$, $F(G_{\mathbf{y}}) = F(G_{\mathbf{x}})$ and $\mathbf{y} \in \mathbf{V}_{n+1}$ or the property $F(G_{\mathbf{y}}) > F(G_{\mathbf{x}})$. In the latter case a contradiction with the choice of $G_{\mathbf{x}}$ is obtained. ■

Lemma 2.7. *Let d, m, n, j be integers $0 < m \leq \binom{n}{2}$, $1 \leq j \leq n$, $1 \leq d \leq n-1$, $n \geq 1$. A sequence of nonnegative integers $(d_1, d_2, \ldots, d_n)$ such that $d_1 = \ldots = d_j = d$; $d_{j+1} = \ldots = d_n = d-1$ and $d_1 + \ldots + d_n = 2m$ is graphical.*

Proof is by induction on n. An idea of Havel and Hakimi (in [4]) can be used. Details are left to the reader. ■

Theorem 1. *Let $p = \lceil \frac{2m}{n} \rceil$ and $q = np - 2m$ then*

$$f(n, m) = np^2 - 2qp + q. \tag{7}$$

Proof. If $G \in \mathcal{G}(n, m)$ with $F(G) = f(n, m)$ then $\Delta(G) - \delta(G) \leq 1$ where $\Delta = \Delta(G)$ and $\delta = \delta(G)$ is the maximum and minimum degree of G respectively. For the contrary suppose that G has vertices v_i and v_k such that $d_i > d_k + 1$. Then there exists a vertex v_j such that $v_i v_j \in E$ and $v_j v_k \notin E$. Then by Lemma 2.1 there exists a graph G^* with $F(G^*) = F(G) + 2(d_k - d_i) + 2 < F(G)$, a contradiction. This means that G contains x vertices of degree Δ and $n - x$ vertices of degree $\Delta - 1$ and therefore

$$x\Delta + (n - x)(\Delta - 1) = 2m \quad \text{e.i.} \quad \Delta = \frac{2m}{n} + 1 - \frac{x}{n}.$$

Table

The form of $\mathbf{z}$	The form of $\mathbf{x}$	Conditions on k,s,j	The form of $\mathbf{y}$	$F(G_{\mathbf{y}}) - F(G_{\mathbf{x}})$
II^*	$(i(1), o(k), i(s))$	$k = 1 \vee s = 1$	$\mathbf{x} \in \mathbf{V}_{n+1}$	0
		$1 < k < s + 1$	$(o(k-1), i(k), o(1), i(s-k+1)) \in \mathbf{V}_{n+1}$	0
		$1 < k = s + 1$	$(o(k-1), i(k), o(1)) \in \mathbf{V}_{n+1}$	0
		$1 < k = s + 2$	$(o(k-1), i(k)) \in \mathbf{V}_{n+1}$	0
		$k > s + 2 \wedge s = = 2$	$(i(1), o(k-1), i(1), o(2)) \in \mathbf{V}_{n+1}$	0
		$k > s + 2 \wedge s = = 3$	$(i(1), o(k-3), i(1), o(5)) \in \mathbf{V}_{n+1}$	6
		$k > s + 2 \wedge s > > 3$	$(i(1), o(k-3), i(1), o(s+1), i(s-3), o(1)) \notin \mathbf{V}_{n+1}$	6
III^*	$(i(1), o(k), i(s) o(1))$	$s = 1$	$\mathbf{x} \in \mathbf{V}_{n+1}$	0
		$k = 1$	$(o(1), i(s+2)) \in \mathbf{V}_{n+1}$	$2s > 0$
		$2 \leq k < s + 1$	$(o(k-1), i(k-1) o(1), i(s-k+3)) \in \mathbf{V}_{n+1}$	$2(1 + s - k) > > 0$
		$k \geq s + 1 \wedge s = = 2$	$(i(1), o(k-2) i(1), o(4)) \in \mathbf{V}_{n+1}$	4
		$k \geq s + 1 \wedge s \geq \geq 3$	$(i(1), o(k-s), i(1), o(s+1), i(s-2), o(1)) \notin \mathbf{V}_{n+1}$	4
IV^*	$(i(1).o(k), i(s), o(1), i(j))$	$k < j$	$(o(k), i(s+k+1), o(1), i(j-k)) \in \mathbf{V}_{n+1}$	$2ks > 0$
		$k = j$	$(o(k), i(s+k+1), o(1)) \in \mathbf{V}_{n+1}$	$2ks > 0$
		$k = j + 1$	$(o(k), i(s+j+2)) \in \mathbf{V}_{n+1}$	$2ks > 0$
		$j + 1 < k < < j + s + 1$	$(o(k-1), i(k-j-1), o(1), i(3+s+2j-k)) \in \mathbf{V}_{n-1}$	$2(j+1) \cdot \cdot (1 + s + j - -k) > 0$
		$k \geq s + j + 1 \wedge \wedge s = j = 1$	$(i(1), o(k-1), i(1), o(3)) \in \mathbf{V}_{n+1}$	2
		$k \geq s + j + 1 \wedge \wedge j > s = 1$	$(i(1), o(k-j), i(1), o(j+2), i(j-1)) \notin \mathbf{V}_{n+1}$	$2j > 0$
		$k \geq s + j + 1 \wedge \wedge s > j = 1$	$(i(1), o(k-s), i(1), o(s+1), i(s-1), o(1)) \notin \mathbf{V}_{n+1}$	2
		$k \geq s + j + 1 \wedge \wedge s > 1 \wedge j > 1$	$(i(1), o(1+k-j-s), i(1), o(j+s), i(s-1), o(1), i(j-1)) \notin \mathbf{V}_{n+1}$	$2j > 0$

Since $x \geq 1$ we have

$$\Delta = \left\lceil \frac{2m}{n} \right\rceil = p \quad \text{and} \quad x = n - q \tag{8}$$

Since $f(n,m) = F(G) = x\Delta^2 + (n-x)(\Delta-1)^2$ we have immediately (7). ∎

Remark 1. An extremal graph for $f(n,m)$ exists and has x vertices of degree p and $n-x$ vertices of degree $p-1$ with $px + (n-x)(p-1) = 2m$. Its existence is guaranteed by Lemma 2.7.

Theorem 2. *Let* $k = \left\lfloor \frac{1}{2}(2n-1-\sqrt{(2n-1)^2-8m}) \right\rfloor$, $r = \frac{1}{2}(2m - k(2n-k-1))$, $a = \lceil \frac{1}{2}(1+\sqrt{1+8m}) \rceil$ *and* $b = \frac{1}{2}(a^2-a-2m)$. *Then* $F(n,m) = \max\{k(n-1)^2 + (k+r)^2 + r(k+1)^2 + (n-k-r-1)k^2;\ (a-b-1)(a-1)^2 + b(a-2)^2 + (a-b-1)^2\}$.

Proof. By Lemma 2.6 there is $\mathbf{x} \in \mathbf{V}_n$ such that $G_\mathbf{x} \in \mathcal{G}(n,m)$ and $F(G_\mathbf{x}) = F(n,m)$. Two cases are to be considered. Case 1. Let $\mathbf{x}$ be one of the form I-VI. For all these forms there is $\mathbf{x} = (i(k), o(s), i(l), o(j))$ with nonnegative integers k, s, j, l, $l = 0$ or 1, such that $k+s+l+j = n-1$. The corresponding graph $G_\mathbf{x}$ has k vertices of degree $n-1$, s vertices of degree k, l vertices of degree $k+j+1$ and $j+1$ vertices of degree $k+l$. This means that $2m = k(n-1) + sk + l(k+j+1) + (j+1)(k+l)$ and so $k(n-1) + (n-k)k \leq 2m$.

It is left to the reader to show that k is the largest integer fulfilling the last inequality. That is why $k = \lfloor \frac{1}{2}(2n-1-\sqrt{(2n-1)^2-8m}) \rfloor$ and $2l(j+1) = 2m - k(n-1) - (n-k)k = 2m - k(2n-k-1) = 2r$. Because of $s = n-k-l-j-1$ and by (6) we have $F(G_\mathbf{x}) = k(n-1)^2 + (n-k-j-l-1)k^2 + l(k+j+1)^2 + (j+1)(k+l)^2$. Since $l = 0$ implies $r = 0$ and $l = 1$ means $r = l+1$ we can easily obtain the first part of required formula for $F(n,m)$.

Case 2. Let x be one of the form I*–VI*. For all these cases $\mathbf{x} = (o(d), i(s), o(l), i(j))$ with nonnegative integers d, s, l, j, $l = 0$ or 1 and $d+s+l+j = n-1$. The corresponding graph $G_\mathbf{x}$ has d isolated vertices, s vertices of degree $n-d-1$, l vertices of degree s and $j+1$ vertices of degree $n-d-1-l$. This means that $G_\mathbf{x}$ can be expressed by $\overline{K}_d \cup G_a$ where G_a is a graph K_a with $a = n-d$ or a graph obtained from K_a by deleting $j+1$ edges incident with a vertex of K_a. Therefore $a(a-1) \geq 2m$ and a is the smallest integer satisfying the last inequality. That is $a = \lceil \frac{1}{2}(1+\sqrt{1+8m}) \rceil$ and $2(j+1) = a^2 - a - 2m = 2b$. Since for $l = 0$ there is $b = 0$ we have second part of $F(n,m)$. Let $l = 1$, then $2m = s(n-d-1)+s+(j+1)(n-d-2) = s(a-1)+s+(j+1)(a-2)$, $2m = s(a-1)+s+(a-s-1)(a-2)$ that is $s = a-b-1$.

By (6) we have

$$F(G_x) = s(n-d-1)^2 + s^2 + (j+1)(n-d-2)^2 =$$
$$(a-b-1)(a-1)^2 + (a-b-1)^2 + b(a-2)^2. \blacksquare$$

Remark 2. From the proof of Theorem 2 it follows that an extremal graph for $F(n,m)$ exists and has k vertices of degree $(n-1)$, one vertex of degree $k+r$, r vertices of degree $k+1$ and $(n-k-r-1)$ vertices of degree k or it has d isolated vertices, $d = n-a$, $a-b-1$ vertices of degree $a-1$, one vertex of degree $a-b-1$ and b vertices of degree $a-2$.

3. The original problem

In this section we discuss the numbers $w(n,m)$ and $W(n,m)$. For the first one we have

Theorem 3. *Let $p = \lceil \frac{2m}{n} \rceil$ and $q = np - 2m$. Then*

$$w(n,m) = \begin{cases} 2p-1 & \text{if } (n-q)p \le m \text{ and } p \le q, \\ 2p & \text{if } (n-q)p > m \text{ or } p > q. \end{cases}$$

Proof. By Theorem 1 we have

$$w(n,m) \ge \left\lceil \frac{np^2 - 2pq + q}{m} \right\rceil = 2p - 2 + \left\lceil \frac{n-q}{m} p \right\rceil. \tag{9}$$

Let $(n-q)p \le m$ then $0 < \frac{n-q}{m} p \le 1$ and therefore by (9)

$$w(n,m) \ge 2p - 1.$$

If $p \le q$ then there exists a graph G with the maximum edge weight $2p-1$. We construct the graph $G = (V,E)$ with the vertex set $V = V_1 \cup V_2$,

$V_1 \cap V_2 = \emptyset$, such that $|V_1| = n-q$, $|V_2| = q$ and such that every vertex of V_1 is p valent and every vertex of V_2 is $p-1$ valent. Let $V_1 = \{v_1, v_2, \ldots, v_{n-q}\}$ and $V_2 = \{z_1, \ldots, z_q\}$.

Let $x = \frac{2m-(n-q)p}{q}$. By Lemma 2.7 there exists a graph $G_2 = (V_2, E_2)$ having c vertices $z_1, \ldots, z_c$ of degree $\lfloor x \rfloor$ and the rest $q-c$ vertices of degree $\lceil x \rceil$ such that

$$c\lfloor x \rfloor + (q-c)\lceil x \rceil = 2m - (n-q)p.$$

To obtain the graph G required every vertex v_j of V_1, $j = 1, 2, \ldots, n-q$, is joined by an edge with the vertices $z_{(j-1)p+1}, z_{(j-1)p+2}, \ldots, z_{jp}$ of the graph G_2. (Indices are taken modulo q.)

Let $p > q$. Let there exists a graph $G^* \in \mathcal{G}(n,m)$ with $\deg u + \deg v \leq 2p-1$ for every edge uv of the graph G^*. Put $\Delta(G^*) = p+t$, then $0 \leq t \leq n-1-p$. If G^* has r vertices of degrees at most $p-1-t$, then other vertices of G^* have degrees at most $p+t$ and we have

$$(n-r)(p+t) + r(p-t-1) \geq 2m.$$

Since $np = 2m + q$ we get after routine manipulations

$$r \leq \frac{q+nt}{2t+1} = \frac{\frac{n}{2}(2t+1) - \frac{n}{2} + q}{2t+1} = \frac{n}{2} + \frac{q - \frac{n}{2}}{2t+1}. \tag{10}$$

If $q - \frac{n}{2} \geq 0$ then $\frac{q-\frac{n}{2}}{2t+1} \leq q - \frac{n}{2}$ and therefore by (10) $r \leq q < p$. If $q - \frac{n}{2} < 0$ then $r < \frac{n}{2}$. As $q = np - 2m$ we have $2(n-q)p \leq np - q$. This implies $q(2p-1) \geq pn > nq$ that is $p > \frac{n}{2} > r$. In both cases we have obtained $p > r$ which means that G^* contains an edge e joining a vertex of degree $\Delta(G^*)$ with a vertex of degree at least $p-t$. This provides a contradiction with the choice of the graph G^* because $w(e) \geq 2p$.

Let $(n-q)p > m$ then $1 < \frac{(n-q)p}{m} \leq \frac{2m}{m} = 2$ and by (10) we have $w(n,m) \geq 2p$. To verify the equality, it is sufficient to find a graph having the maximum weight of edges $2p$. Such graph G exists and it is mentioned in the proof of Theorem 1 with $F(G) = f(n,m)$. (Or Lemma 2.7 can be used with $d = \lceil \frac{2m}{n} \rceil$ and $dj + (d-1)(n-j) = 2m$.) ■

It seems to be difficult to determine the value $W(n,m)$. By the list of six vertex graphs in [4] it can be find out that $W(6,6) = W(6,7) = W(6,8) = 6$, $W(6,9) = W(6,11) = 7$, $W(6,10) = W(6,12) = 8$.

It is easy to see that the weight of any edge of a graph $G \in \mathcal{G}(n,m)$ cannot be larger than $m+1$. Because of the graph $K_{1,m} \cup \overline{K}_{n-m-1}$ we have $W(n,m) = m+1$ for every $1 \le m < n$. Other known results are stated in

Theorem 4. *Let $m = \binom{n}{2} - r$ and let $0 \le r < n-1$. Then*
(i) $W(n,m) = 2n-2$ for $r = 0$ and $W(n,m) = 2n-3$ for $r = 1$;
(ii) $W(n,m) = 2n-4$ for $2 \le r \le \lfloor \frac{n}{2} \rfloor$ or $r = 3$;
(iii) $W(n,m) = 2n-5$ for $\lfloor \frac{n}{2} \rfloor < r \le \lceil \frac{n+2}{2} \rceil$ or $r = 6$;
(iv) $W(n,m) = 2n-6$ in all other cases.

Proof. (i) is trivial. In the case (ii) it is sufficient to realize that if a graph $G \in \mathcal{G}(n,m)$ contains a vertex of degree not exceeding $n-3$ then it contains an edge of the weight less or equal $2n-4$. If r independent edges or edges of a triangle (if $r = 3$) are removed from the graph K_n we obtain suitable graph. Similar arguments have to be used in the cases (iii) and (iv). A suitable graph in the case (iii) is obtained by removing $r-3$ independent edges and edges of an independent triangle from K_n. In the case (iv) edges of a cycle of the length r are deleted from K_n. ■

A difficulty in determining $W(n,m)$ can also be in non unambiguous existence of the graphs realizing $W(n,m)$. For example there are two graphs realizing the weight $W\left(n, \binom{n}{2} - 2\right)$, a graph K_n with two independent edges removed and a graph K_n with two neighbouring edges deleted.

We are able to prove the next result.

Theorem 5. *Let a, b be integers defined in Theorem 2, let $h = \left\lceil \frac{1}{2}(2n-1-\sqrt{(2n-1)^2-8m}) \right\rceil$ and let s, t be integers such that $ht + s = m$, $h + t \le n$ and $h(h-3) < 2s \le h(h-1)$. Let $g(n,m)$ be as follows*

$$g(n,m) = \begin{cases} 2a-2 & \text{if } b = 0; \\ 2a-3 & \text{if } b = 1; \\ 2a-4 & \text{if } 2 \le b \le \lfloor \frac{a}{2} \rfloor \text{ or } b = 3; \\ 2a-5 & \text{if } \lfloor \frac{a}{2} \rfloor < b \le \lceil \frac{a+2}{2} \rceil \text{ or } a = 8 \text{ and } b = 6; \\ 2a-6 & \text{in all other cases.} \end{cases}$$

Then

$$\max\left\{h + t + \left\lfloor \frac{2s}{h} \right\rfloor, \; g(n,m)\right\} \le W(n,m) \le \left\lfloor \frac{F(n,m)}{m} \right\rfloor. \qquad (11)$$

Proof. Let G_h be a graph with h vertices, s edges where the vertices have only degrees $\lceil \frac{2s}{h} \rceil$ or $\lfloor \frac{2s}{h} \rfloor$. By Lemma 2.7 such graph there exists. Now it is easy to see that the graph $G^* = (\overline{K}_t \oplus G_h) \cup \overline{K}_{n-h-t}$ belongs to $\mathcal{G}(n,m)$ and minimum weight of its edges is $h + t + \lfloor \frac{2s}{h} \rfloor$.

It is easy to see that $0 \le b < a - 1$. A graph $\hat{G} \in \mathcal{G}(n,m)$ having the minimum weight of edges equal to $g(n,m)$ can be obtained from the graph K_a by deleting b edges in a suitable way.

The graphs G^* and $\hat{G}$ provide a lower bound for $W(n,m)$. The upper bound in (11) easily follows from Theorem 2. ■

Conjecture. We believe that

$$W(n,m) = \max\left\{h + t + \left\lfloor \frac{2s}{h} \right\rfloor, \quad g(n,m)\right\}.$$

The proof of the next theorem is easy.

Theorem 6. *Let*

$$z(n,m) = \begin{cases} 2 & \text{for } m \le 1 + \binom{n-2}{2} \\ 1 + m - \binom{n-2}{2} & \text{for } m > 1 + \binom{n-2}{2} \end{cases}$$

and let

$$Z(n,m) = \begin{cases} 1 + m & \text{for } m < 2n - 3 \\ 2(n-1) & \text{for } m \ge 2n - 3. \end{cases}$$

For every edge e of a graph $G \in \mathcal{G}(n,m)$ there is

$$Z(n,m) \le w(e) \le Z(n,m).$$

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References

[1] O. V. Borodin, Computing Light Edges in Planar Graphs, in: *Topics in Combinatorics and Graph Theory* (eds.: R. Bodendiek and R. Henn) Physica-Verlag, Heidelberg, 1990, 137–144.

[2] B. Grünbaum, Acyclic colorings of planar graphs, *Israel J. Math.* **14**(1973), 390–408.

[3] B. Grünbaum, G. S. Shephard, Analogs for tilings of Kotzig's theorem on minimal weight of edges, *Ann. Discrete Mat.* **12**(1982), 129–140.

[4] F. Harary, *Graph Theory,* Addison-Wesley, Massachusetts, 1969.

[5] J. Ivančo, The weight of a graph, *Proceedings of the 4th Czechoslovak Symposium on Combinatorics, Prachatice,* 1990, (to appear).

[6] E. Jucovič, Strengthening of a theorem about 3-polytopes, *Geometriae Dedicata* **13**(1974), 233-237.

[7] A. Kotzig, Contribution to the theory of Eulerian polyhedra, *Mat. Čas.* **5**(1955), 101–103 (in Slovac, Russian summary).

[8] J. Zaks, Extending Kotzig's theorem, *Israel J. Math.* **45**(1983), 281–296.

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