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The Long Club (♣)

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ABSTRACT

For any ordinal α let $♣(\alpha)$ denote the statement "with each $\nu \in \alpha$ with $\mathrm{cf}(\nu) = \omega$ we can associate a countable set S_ν such that $\cup S_\nu = \nu$ and for every $X \in [\alpha]^{\omega_1}$ there is a ν with $S_\nu \subset X$". Thus $♣ = ♣(\omega_1)$. We show that (i) $♣ \longrightarrow ♣(\alpha)$ for all $\alpha \in \omega_2$; (ii) $♣ + \square \longrightarrow ♣(\omega_2)$; (iii) $♣(\omega_2 \cdot \omega_1)$ is false; (iv) $♣ + \neg♣(\omega_2)$ is consistent (modulo some large cardinals).

In [4] (cf. also [1]) the combinatorial principle ♣ (club) was introduced in the form "with each limit ordinal $\nu \in \omega_1$ we can associate a cofinal subset S_ν such that for every $X \in [\omega_1]^{\omega_1}$ there is a ν with $S_\nu \subset X$." The sequence $\langle S_\nu : \nu \in L_1 \rangle$ (here, of course, L_1 denotes the set of all limit ordinals in ω_1 and in what follows we denote by $L_1(\alpha)$ the set of all $\nu \in \alpha$ with $\mathrm{cf}(\nu) = \omega$) is then called a ♣-sequence. In this paper we investigate the following natural "lengthening" of ♣:

Definition 1. For any ordinal α we let $♣(\alpha)$ denote the statement that with every limit ordinal $\nu \in \alpha$ with $\mathrm{cf}(\nu) = \omega$ we can associate a cofinal subset S_ν in such a way that for every uncountable set $X \subset \alpha$ there is a ν with $S_\nu \subset X$. The sequence $\langle S_\nu : \nu \in L_1(\alpha) \rangle$ is then called a $♣(\alpha)$-sequence.

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The first natural question concerning this notion is, of course, for what ordinals $\alpha > \omega_1$ does $\clubsuit$, the shortest possible club, imply $\clubsuit(\alpha)$? It is obvious that $\clubsuit \longrightarrow \clubsuit(\alpha)$ for $\alpha < \omega_1 \cdot \omega_1$, but the same implication for $\omega_1 \cdot \omega_1$ is already not trivial. This, and much more, will follow from the next lemma.

Lemma 1. *Let $\alpha \in \omega_2$ with $\mathrm{cf}(\alpha) = \omega_1$ and $C \subset \alpha$ be a closed unbounded subset of α of order type $\mathrm{tp}\ C = \omega_1$. If $\langle S_\nu : \nu \in L_1 \rangle$ is a $\clubsuit$-sequence and $h: \omega_1 \longrightarrow \alpha$ is a bijection then for every cofinal subset $X \subset \alpha$ there is a $\nu \in L_1$ such that $\cup h[S_\nu] = \gamma_\nu$ and $h[S_\nu] \subset X$, where γ_ν is the ν^{th} element of C.*

Proof. Let us start by recalling that if $\langle S_\lambda : \lambda \in L_1 \rangle$ is a $\clubsuit$-sequence then for every $Y \in [\omega_1]^{\omega_1}$ we actually have that the set

$$\{\lambda \in L_1 : S_\lambda \subset Y\}$$

is stationary in ω_1. Indeed, this follows immediately from the easy observation that for every closed and unbounded set $B \subset \omega_1$ there is a subset $Z \subset Y$ with $Z' = Y' \cap B$ (for A a set of ordinals we let A' denote the set of limit points of A).

Without loss of generality we may assume that the set $X \subset \alpha$ has order type ω_1, i.e. for every $\beta \in \alpha$ we have $|X \cap \beta| < \omega_1$.

Let us now put

$$B = \{\beta \in \omega_1 : \forall \nu \in \beta (h(\nu) \in \gamma_\beta)\}$$

(recalling that $C = \{\gamma_\beta : \beta \in \omega_1\}$ is the increasing enumeration of C). Clearly, B is a closed unbounded set in ω_1. Thus, in view of our introductory remark, the set

$$D = \{\beta \in B \cap L_1 : S_\beta \subset h^{-1}[X]\}$$

is stationary in ω_1. By the definition of B we have $h[S_\beta] \subset \gamma_\beta$ for every $\beta \in B$, consequently it suffices to prove that there is some $\nu \in D$ with $\cup h[S_\nu] = \gamma_\nu$.

Assume, on the contrary, that for every $\delta \in D$ we have $\cup h[S_\delta] < \gamma_\delta$. Since δ is limit and C is closed, we can then also choose an $f(\delta) < \delta$ such that $\cup h[S_\delta] < \gamma_{f(\delta)}$ holds as well.

By the pressing down lemma there is an uncountable set $E \in [D]^{\omega_1}$ and an ordinal $\varrho \in \omega_1$ such that $f(\delta) = \varrho$ for every $\delta \in E$. But then we have

$$\cup\{h[S_\delta] : \delta \in E\} \subset X \cap \gamma_\varrho ,$$

consequently $|X \cap \gamma_\varrho| = \omega_1$, because h is one-to-one, contradicting our assumption. This concludes the proof. ■

The next result now follows easily.

Theorem 1. *♣ implies ♣(α) for every $\alpha \in \omega_2$.*

Proof. We use induction on $\alpha \in \omega_2$. So assume ♣(β) holds for each $\beta < \alpha$ and, of course, $\alpha > \omega_1$.

If $\alpha = \omega_1 \cdot \gamma + \beta$ with $0 < \beta < \omega_1$, then a ♣$(\omega_1 \cdot \gamma)$-sequence can be trivially extended to a ♣(α)-sequence, for $X \in [\alpha]^{\omega_1}$ implies $|X \cap \omega_1 \cdot \gamma| = \omega_1$ as well.

Thus we may assume that α is of the form $\omega_1 \cdot \gamma$. If $\gamma = \beta + 1$ then we get a ♣(α)-sequence from the ♣$(\omega_1 \cdot \beta)$-sequence and the ♣-sequence "translated" to the final segment $[\omega_1 \cdot \beta, \alpha)$. Similarly, if $\mathrm{cf}(\gamma) = \omega$ then we pick a cofinal ω-sequence $\{\gamma_n : n \in \omega\}$ in γ and for each $n \in \omega$ we fix a "translated" ♣$(\mathrm{tp}(\omega_1 \cdot (\gamma_{n+1} - \gamma_n)))$-sequence $\mathcal{S}_n$ in the interval $[\omega_1 \cdot \gamma_n, \omega_1 \cdot \gamma_{n+1})$. Clearly, then $\cup\{\mathcal{S}_n : n \in \omega\}$ yields a ♣(α)-sequence.

Finally, if $\alpha = \omega_1 \cdot \gamma$ with $\mathrm{cf}(\gamma) = \omega_1$ then fix an ω_1-type closed unbounded set B in γ with $B = \{\beta_\nu : \nu \in \omega_1\}$ and set $C = \{\omega_1 \cdot \beta_\nu : \nu \in \omega_1\}$, moreover let $h : \omega_1 \longrightarrow \alpha$ be a bijection between ω_1 and α. Now, for each $\nu \in \omega_1$ fix a translated ♣$(\mathrm{tp}(\gamma_{\nu+1} \setminus \gamma_n))$-sequence $\mathcal{S}_\nu$ inside the interval $[\gamma_\nu, \gamma_{\nu+1})$, where $\gamma_\nu = \omega_1 \cdot \beta_\nu$. Then, for every limit ordinal $\nu \in L_1, \gamma_\nu$ the ν^{th} element of C is of cofinality ω and we are free to associate with γ_ν the set $h[S_\nu]$ if $\cup h[S_\nu] = \gamma_\nu$ happens to be true. (If $\cup h[S_\nu] \neq \gamma_\nu$ then we can choose the countable cofinal set in γ_ν arbitrarily.) Since for every $X \in [\alpha]^{\omega_1}$ either there is a $\mu \in \omega_1$ with $|X \cap [\gamma_\mu, \gamma_{\mu+1})| = \omega_1$ or $\cup X = \alpha$, the sequence manufactured above is a ♣(α)-sequence, applying Lemma 1 to h, C and $\langle \mathcal{S}_\nu : \nu \in L_1 \rangle$. The theorem is now proven. ■

The following problem to ask now is whether ♣(ω_2), this is what we call the long club, is consistent or even implied by ♣? (Note that if $\alpha < \beta$ then ♣(β) clearly implies ♣(α).) We have two results that yield positive answers to the above questions, however before turning to them we first formulate an easy but perhaps surprising non-existence result.

Theorem 2. *♣$(\omega_2 \cdot \omega_1)$ is false.*

Proof. For each $\alpha \in \omega_2$ let us put

$$X_\alpha = \{\omega_2 \cdot \nu + \alpha : \nu \in \omega_1\} .$$

Then $X_\alpha \in [\omega_2 \cdot \omega_1]^{\omega_1}$ and $X'_\alpha = \{\omega_2 \cdot \nu : \nu \in L_1\}$, moreover $X_\alpha \cap X_\beta = \emptyset$ if $\alpha \neq \beta$. Now if $\lambda \in \omega_2 \cdot \omega_1$ is a limit ordinal and a cofinal subset $S \subset \lambda$ is contained in X_α then $\lambda \in X'_\alpha$ hence $\lambda = \omega_2 \cdot \nu$ for some $\nu \in L_1$. Thus if $\clubsuit(\omega_2 \cdot \omega_1)$ were true then, since there are ω_2-many X_α'-s and only ω_1-many $\nu \in L_1$'s, we had two distinct $\alpha, \beta \in \omega_2$ and a $\nu \in L_1$ such that the set S associated with $\omega_2 \cdot \nu$ by $\clubsuit(\omega_2 \cdot \omega_1)$ would be contained in both X_α and X_β, contradicting $X_\alpha \cap X_\beta = \emptyset$. ■

Let us note, on the other hand, that $\clubsuit(\omega_2)$ clearly implies $\clubsuit(\alpha)$ for every $\alpha < \omega_2 \cdot \omega_1$.

Now we turn to the consistency results concerning $\clubsuit(\omega_2)$, the long club.

Theorem 3. *If $\clubsuit$ and $\Box$ are both true then so is $\clubsuit(\omega_2)$. In particular, $\clubsuit(\omega_2)$ holds in L.*

Proof. Let us recall (see [1]) that $\Box$ says: there exists a sequence $\langle C_\xi : \xi \in L_2 \rangle$ (here L_2 denotes the set of all limit ordinals in ω_2) such that (i) for every $\xi \in L_2$ the set C_ξ is closed and unbounded in ξ with $\mathrm{tp}\,(C_\xi) \leq \omega_1$, and (ii) if $\eta \in C'_\xi$ then $C_\eta = \eta \cap C_\xi$.

Claim. For all $\xi \in L_2$ we can fix a bijection $h_\xi : \omega_1 \longrightarrow \omega_1 \cdot \xi$ between ω_1 and $\omega_1 \cdot \xi$ in such a way that if $\alpha \in C'_\xi \cap C'_\eta$ then

$$h_\xi^{-1} \restriction \omega_1 \cdot \alpha = h_\eta^{-1} \restriction \omega_1 \cdot \alpha \; .$$

Proof of claim. Let $C_\xi = \{\gamma_\nu^{(\xi)} : \nu \in \tau_\xi\}$ be the increasing enumeration of C_ξ and $\omega_1 = \cup\{A_\nu : \nu \in \omega_1\}$ be a partition of ω_1 into ω_1-many disjoint sets of size ω_1.

For every $\nu \in \tau_\xi$ we may then fix a bijection $h_\nu^{(\xi)} : A_\nu \longrightarrow [\omega_1 \cdot \gamma_\nu^{(\xi)}, \omega_1 \cdot \gamma_{\nu+1}^{(\xi)})$ and then set

$$h_\xi = \cup\{h_\nu^{(\xi)} : \nu \in \tau_\xi\} \; .$$

Now it is clear from (ii) that if $\alpha \in C'_\xi \cap C'_\eta$ then

$$h_\xi^{-1} \restriction \omega_1 \cdot \alpha = h_\alpha^{-1} = h_\eta^{-1} \restriction \omega_1 \cdot \alpha \; ,$$

hence the claim is proven. ■

Next, given a $\clubsuit$-sequence $\langle S_\lambda : \lambda \in L_1 \rangle$ we first put for any limit ordinal of the form $\omega_1 \cdot \alpha + \beta$ with $0 < \beta < \omega_1$ (i.e. $\beta \in L_1$)

$$T_{\omega_1 \cdot \alpha + \beta} = \omega_1 \cdot \alpha + S_\beta \; ,$$

and if $\alpha \in C'_\xi$ with $\mathrm{cf}(\xi) = \omega_1$ and $\mathrm{cf}(\alpha) = \omega$, i.e. $\alpha = \gamma_\nu^{(\xi)}$ for some $\nu \in L_1$, then we set

$$T_{\omega_1 \cdot \alpha} = h_\xi[S_\nu]$$

provided that

$$\cup h_\xi[S_\nu] = \omega_1 \cdot \alpha = \omega_1 \cdot \gamma_\nu^{(\xi)} .$$

The above claim insures that this definition of $T_{\omega_1 \cdot \alpha}$ does not depend on ξ.

Finally, if $\omega_1 \cdot \alpha$ with $\alpha \in L_1(\omega_2)$ is not like above we may choose $T_{\omega_1 \cdot \alpha}$ arbitrarily.

Now it remains to check that $\langle T_\lambda : \lambda \in L_1(\omega_2)\rangle$ is a $\clubsuit(\omega_2)$-sequence. So let $X \subset \omega_2$ be uncountable, without loss of generality we may assume that $\mathrm{tp}\ X = \omega_1$. If there is some α with

$$|X \cap [\omega_1 \cdot \alpha,\ \omega_1 \cdot (\alpha + 1))| = \omega_1$$

then clearly there is a $\beta \in L_1$ with

$$T_{\omega_1 \cdot \alpha + \beta} \subset X \cap [\omega_1 \cdot \alpha,\ \omega_1 \cdot (\alpha + 1)),$$

hence we may assume that $\cup X = \omega_1 \cdot \xi$ with $\mathrm{cf}(\xi) = \omega_1$. But then Lemma 1 applied to h_ξ, the closed unbounded set $D_\xi = \{\omega_1 \cdot \gamma_\nu^{(\xi)} : \nu \in \omega_1\} \subset \omega_1 \cdot \xi$, and the $\clubsuit$-sequence $\langle S_\nu : \nu \in L_1\rangle$ yields an $\alpha \in C'_\xi$ such that

$$T_{\omega_1 \cdot \alpha} = h_\xi[S_\nu] \subset X .$$

This completes the proof. ■

The following result yields the consistency of the long club, $\clubsuit(\omega_2)$, via "simple" forcing.

Theorem 4. *If $V \models CH$ then there is an ω_1-closed notion of forcing P with the $\omega_2 - CC$ such that $V^P \models \clubsuit(\omega_2)$.*

Proof. The natural partial order P consisting of countable fragments of a $\clubsuit(\omega_2)$-sequence will work. In other words, the elements $p \in P$ are functions whose domain $D(p)$ is a countable subset of $L_1(\omega_2)$ and for each $\alpha \in D(p)$ the value $p(\alpha)$ is a countable cofinal subset of α. Moreover, for $p, q \in P$ we have $p \leq q$ iff $p \supset q$.

The ω_1-closedness of P is obvious and the ω_2-CC follows from CH by a standard Δ-system and counting argument. Thus, cardinals and cofinalities in V^P are preserved. It is also obvious that for any $\alpha \in L_1(\omega_2)$ the set

$$D_\alpha = \{p \in P : \alpha \in D(p)\}$$

of conditions is dense in P, hence if G is P-generic over V and we put

$$S_\alpha = \cup G(\alpha) ,$$

then the sequence $\langle S_\alpha : \alpha \in L_1(\omega_2)\rangle$ is defined everywhere.

Next, we show that this sequence is a $\clubsuit(\omega_2)$-sequence in V^P. To see this, let $\dot{X}$ be a P-name such that

$$1_P \Vdash \text{"}\dot{X} \subset \omega_2 \quad \text{and} \quad \operatorname{tp}(\dot{X}) = \omega_1\text{"} .$$

Clearly, we shall be done if we can show that every condition $p \in P$ has an extension q that forces $S_\eta \subset \dot{X}$ for some $\eta \in L_1(\omega_2)$.

To see this, we first define a decreasing sequence of conditions below p as follows. Let p_0 be an extension of p such that for some $\xi \in \omega_2$

$$p_0 \Vdash \cup \dot{X} = \xi .$$

Clearly, then $\operatorname{cf}(\xi) = \omega_1$. Now, if for some $n \in \omega$ we have already defined p_n then, since $|D(p_n)| \leq \omega$, we can find an ordinal η_n and a condition $p_{n+1} \leq p_n$ such that $\cup D(p_n) \cap \xi < \eta_n < \xi$ and $p_{n+1} \Vdash \eta_n \in \dot{X}$. Clearly, we can do this in such a way that $\eta_n < \eta_{n+1}$ for all $n \in \omega$. Then $p_\omega = \cup\{p_n : n \in \omega\} \in P$ satisfies

$$p_\omega \Vdash y \subset \dot{X}$$

for the set $y = \{\eta_n : n \in \omega\}$, and if we put $\eta = \cup\{\eta_n : n \in \omega\}$ then $\eta \notin D(p_\omega)$. Consequently $q = p_\omega \cup \{\langle \eta, y\rangle\} \in P$ is an extension of p such that $q \Vdash S_\eta \subset \dot{X}$. This completes the proof. ■

Before we turn to the question of the joint consistency of $\clubsuit + \neg\clubsuit(\omega_2)$ it will be useful to explicitly formulate a consequence of $\clubsuit(\alpha)$ that is most conveniently done in partition theoretic terms. Let us recall that the symbol

$$\alpha \longrightarrow (\text{top } \beta)_\gamma$$

means the following: for every partition $k : \alpha \longrightarrow \gamma$ there is a topological, i.e. homeomorphic copy of β in α that is k-homogeneous.

More precisely, this means that there is a set $Y \subset \alpha$ such that $|k[Y]| = 1$ and Y as a subspace of α in the order topology is homeomorphic to ω_1 with its natural (order) topology.

Theorem 5. *If ♣(α) holds then*

$$\alpha \not\rightarrow (\text{top } \omega_1)_2 \ .$$

Proof. We define the partition $k\colon \alpha \longrightarrow 2$ by induction on $\nu \in \alpha$. Let $\langle S_\nu : \nu \in L_1(\alpha)\rangle$ be a ♣(α)-sequence. If $\nu \in L_1(\alpha)$ and $k(\mu)$ has been defined for all $\mu \in \nu$ then look at whether S_ν is k-homogeneous (i.e. $|k[S_\nu]| = 1$), and if it is, then define $k(\nu)$ in such a way that $k(\nu) \neq k(\mu)$ for $\mu \in S_\nu$. In all other cases $k(\nu)$ may be defined freely.

Now, assume that X is an uncountable subset of α that is k-homogeneous. By ♣(α) there is a $\nu \in L_1(\alpha)$ with $S_\nu \subset X$, hence by our construction we have $k(\nu) \neq k(\mu)$ for $\mu \in S_\nu$, consequently $\nu \notin X$. This, however, shows that X cannot be a topological copy of ω_1. ■

In view of this result the consistency of ♣$+\neg$♣(ω_2) will follow if we can construct a model in which ♣ holds and $\omega_2 \longrightarrow (\text{top } \omega_1)_2$ is satisfied. By theorem 3 in such a model the principle □ must fail and, as is well-known, large (at least Mahlo) cardinals are needed for that. Our construction below requires much larger cardinals, because we need the consistency of

$$c^+ \longrightarrow (\text{top } \omega_1)_c \ ,$$

which we claim is valid in models of Martin's Maximum (see [2]). Of course, here $c = 2^\omega$, and under MM we have $c = \omega_2$. Also, in [2] it was shown that if MM holds then every stationary set of ω-limits in ω_3 contains a topological copy of ω_1. Clearly, this implies $\omega_3 \longrightarrow (\text{top } \omega_1)\omega_2$.

Theorem 6. *Assume that*

$$V \models c^+ \longrightarrow (\text{top } \omega_1)_c$$

and $P = \mathcal{F}n(c, 2; \omega_1)$ is the standard partial order that adds a Cohen subset of c with countable conditions. Then

$$V^P \models ♣ + \neg ♣(\omega_2) \ .$$

Proof. It is well-known, cf. e.g. [3], that adding a Cohen subset of ω_1 with countable conditions will make ◇ hold in the extension, hence $V^P \models$ ♣ is immediate. It is also known that the 2^ω of V is collapsed to have size ω_1 in V^P and, since $|\mathcal{F}n(c,2;\omega_1)| = c$, $(c^+)^V = \omega_2$ in V^P.

Finally, we show that

$$V^P \models \omega_2 \longrightarrow (\text{top } \omega_1)_{\omega_1} ,$$

consequently

$$V^P \models \neg\clubsuit(\omega_2)$$

as well. Indeed, let $\dot{k}$ be a P-name with

$$1_P \Vdash \dot{k} : c^+ \longrightarrow \omega_1 .$$

Then for every $p \in P$ and $\alpha \in c^+$ there is a condition $p_\alpha \le p$ and an ordinal $\nu_\alpha \in \omega_1$ such that

$$p_\alpha \Vdash \dot{k}(\alpha) = \nu_\alpha .$$

Since $|P| = |P \times \omega_1| = c$, applying the relation $c^+ \longrightarrow (\text{top } \omega_1)_c$ to the assignment $\alpha \mapsto \langle p_\alpha, \nu_\alpha \rangle$ in V, we obtain a topological copy $Y \subset c^+$ of ω_1, and a pair $\langle q, \eta \rangle \in P \times \omega_1$ such that $\langle p_\alpha, \nu_\alpha \rangle = \langle q, \eta \rangle$ for every $\alpha \in Y$. But then q forces that Y is a $\dot{k}$-homogeneous topological copy of ω_1, showing that $\omega_2 \longrightarrow (\text{top } \omega_1)_{\omega_1}$ is indeed valid in V^P. ■

To conclude, let us emphasize that the exact consistency strength of $\clubsuit + \neg\clubsuit(\omega_2)$ remains unknown.

Added in proof: The referee and S. Shelah have both pointed it out to us that by a result in [5] $\clubsuit + \neg\clubsuit(\omega_2)$ is actually equiconsistent with a Mahlo cardinal. Indeed, by Theorem 7.1 on p. 388 in [5] from a Mahlo cardinal one gets the consistency of GCH + every stationary set of ω-limits in ω_2 contains a topological copy of ω_1, hence also of $\omega_2 \longrightarrow (\text{top } \omega_1)_{\omega_1}$. Applying Theorem 6 to this the result follows immediately. We thank the referee and S. Shelah for drawing our attention to this.

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