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Long Stars Specify χ-Bounded Classes

H. A. KIERSTEAD*

ABSTRACT

We investigate a conjecture due independently to Gyárfás and Sumner concerning the chromatic number of graphs which do not contain certain trees as induced subgraphs. In Particular, we show that the tree S_k^3, formed by identifying the end points of k paths on four vertices specifies a weakly χ-bounded class.

1. Introduction

A class of graphs Γ is said to be *χ-bounded* if there exists a function f such that for every $G \in \Gamma$, $\chi(G) \leq f(\omega(G))$, where $\chi(G)$ is the chromatic number of G and $\omega(G)$ is the maximum clique size of G; Γ is *weakly χ-bounded* if there exists a constant c such that $\chi(G) \leq c$ for every triangle-free graph in Γ. For a set of graphs Σ, let $\text{Forb}(\Sigma)$ denote the class of graphs, which do not contain any graph in Σ as an induced subgraph. We say that Σ *specifies* $\text{Forb}(\Sigma)$. If $\Sigma = \{H\}$ we may abuse notation and write $\text{Forb}(H)$.

In this paper we investigate the following beautiful conjecture due independently to Gyárfás [2] and Sumner [8].

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Conjecture 1. *For any tree T, Forb(T) is χ-bounded.*

By the well known theorem of Erdős and Hajnal [1] that there exist graphs with arbitrarily large girth and chromatic number, if Forb(H) is χ-bounded, then H is a forest. It is also easy to show that for any forest F, Forb(F) is χ-bounded iff Forb(T) is χ-bounded for each connected component T of F. Thus the conjecture implies that Forb(H) is χ-bounded iff H is a forest. Gyárfás, Szemerédi, and Tuza [4] proved that radius two trees and paths specify weakly χ-bounded classes. Later Gyárfás [3] proved that all paths, and even brooms, specify χ-bounded classes, and the author and Penrice [6] proved that radius two trees specify χ-bounded classes. Many researchers believe that if one can show a class is weakly χ-bounded then it will be possible to show that it is χ-bounded. The smallest tree D, which is not a broom and does not have radius two, is the path on six vertices with an additional edge joined to one of the two center vertices. Gyárfás [3] asked whether Forb(D) was weakly χ-bounded and the author and Penrice [5] showed that this was the case. In this article we improve this result by showing (Corollary 3) that the tree S_k^3, formed by identifying the end points of k paths on four vertices, specifies a weakly χ-bounded class. One purpose for presenting this result is to demonstrate the use of special subgraphs, which we call templates, for coloring graphs which do not induce certain trees. In the remainder of this section we review our notation.

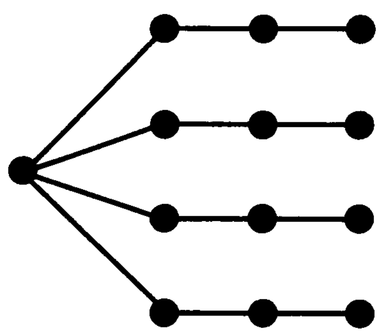

Figure 1. S_4^3

Let P_n denote the path on n vertices. Let S_k^r be the tree formed from k disjoint copies of P_r by joining one end point of each copy with an edge to a new vertex v. See Figure 1. Alternatively, S_k^r is formed by subdividing each of the k edges of the star S_k a total of $r-1$ times. Note that S_k^r is a radius r tree. We call S_k^r a *long star* if $r \geq 2$. Let C_n denote the cycle on n vertices. If C is a cycle we write $C = (c_1, \ldots, c_s)$ to indicate that the vertex set of C is $\{c_1, \ldots, c_s\}$ and c_i is adjacent to c_{i+1}, where addition is modulo s. If $x = c_i$ and $y = c_j$, we write $[x, y]$ for the path $\{c_i, \ldots, c_j\}$ and [x,y) for the path $\{c_i, \ldots, c_{j-1}\}$. Let $K_{t,t}$ denote the complete bipartite graph with t vertices in each part. Let Γ_s be the class Forb$(S_k^r, C_3, C_5, \ldots, C_{2i+1})$, where $s = 2i + 1$. When s is clear from the context, we just write Γ. For a graph $G = (V, E)$, the *open neighborhood* of a set of vertices $X \subset V$ is defined to be $N_G(X) = \{y \in V : xy \in E, \text{ for some } x \in X\}$ the *closed neighborhood* of X is $N_G[X] = N_G(X) \cup X$. When G is clear from the context, we just write $N(X)$ and $N[X]$. With a slight abuse of notation, if a graph plays the role of the set X, then the vertex set of the graph is understood. Also, if $N_G(X)$ (or $N[G]$) is used in the role of a graph, it means the graph induced by the vertex set.

2. Templates

In this section we formulate a general framework for the arguments used in [4], [5], [6], and the next section for coloring graphs based on the notion of template. These techniques were also used by the author, Penrice, and Trotter [7] to find online algorithms for graph coloring. While our formulation accurately reflects the spirit of those arguments, the terminology is new. Let Γ be a class of graphs, which is closed under induced subgraphs, and let T be a tree with a root v. We say that a function $\tau : \Gamma \to \Gamma$ is a T-*template function* for Γ if there exists a function g such that the following three conditions hold, were $\tau(G) = D$.

(1) D is an induced subgraph of G

(2) $\chi(N_G[D]) \leq g(\omega(G))$; and

(3) if $x \in N(D)$, then x is the root of an induced copy of T in $D + x$.

We call the graph $\tau(G)$ a T-*template*. If $|\tau(G)|$ is bounded by b and G is triangle-free, for all $G \in \Gamma$, then we can satisfy (2) by setting $g(2) = b$ (See Lemma 5). A similar idea works when we argue by induction on $\omega(G)$. An interesting innovation of [5], which is also used in the next section, is the use of templates with unbounded size.

Now suppose that τ is a T-template function for Γ and $G \in \Gamma$. We attempt to color G according to the following strategy. Partition the vertices of G into subsets $V_1, \ldots, V_n$ inductively as follows. Let $G_0 = G$. Suppose we have defined G_i and V_j for all $j < i$, so that $G_i \in \Gamma$. Let $D_i = \tau(G_i)$, $V_i = N[D_i]$, $N_i = N(D_i)$, and $G_{i+1} = G_i - V_i$. Note that for $i \neq j$, no vertex of D_i is adjacent to any vertex of D_j.

Each vertex x will receive a two coordinate color $\phi(x) = (\phi_\ell(x), \phi_g(x))$. The first coordinate is chosen from $g(\omega(G))$ local colors; it assures that if two vertices in the same part V_i receive the same color, then they are non-adjacent. The second coordinate is chosen from a set of global colors; it assures that if two vertices from distinct parts V_i and V_j receive the same local color and same global color, then they are non-adjacent. The intention is to use condition (3) to bound the number of global colors used.

The strategy outlined above is the exact strategy which will be employed in the proof of the main theorem. It is a slight modification of the strategy used by Gyárfás, Szemerédi, and Tuza [4] and the author and Penrice [6] to color graphs which do not induce certain radius two trees. In those arguments N_i was the set of vertices which were adjacent to a large number of vertices in D_i. In both papers T was the star S_t with one of its leaves as the root. In [4] it is shown that if a graph, which does not induce a radius two trees, has large chromatic number is terms of its clique size and any integer t, then it induces $K_{t,t}$. This allows the use of a template function τ defined by $\tau(G)$ is an induced $K_{t,t}$ if $(\mathcal{G})$ is sufficiently large and $\tau(G) = G$ otherwise. Clearly, in the triangle-free case, if $x \in N(\tau(G))$, then $\tau(G) + x$ induces S_t. This is no longer true when G is not triangle-free. The main innovation of [6] was to show that complete bipartite graphs could be replaced by certain almost complete multipartite graphs in the definition of τ.

3. The Main Theorem

In this section we prove the following general theorem using the techniques discussed in the previous section.

Theorem 2. *The class of graphs* Γ =Forb$(S_k^r, C_3, C_5, \ldots, C_s)$, *where* $r \leq s = 2i + 1$, *is* χ*-bounded.*

As an immediate consequence of Theorem 2, we obtain:

Corollary 3. *The class of graphs* $\text{Forb}(S_k^3)$ *is weakly* χ*-bounded.*

Let $\tau : \Gamma \to \Gamma$ be defined by $\tau(G) = C$, were C is a minimum odd cycle in G, if $\chi(G) \geq 3$ and $\tau(G) = G$, if $\chi(G) \leq 2$. The next three lemmas will show that τ is a P_{r+1}-template function. Clearly τ is well defined and satisfies (1). The first lemma is an easy technical result which will be used several times.

Lemma 4. *Suppose C is a minimum odd cycle in a graph G. Then the distance d from c_i to c_j in C is at most the distance d' from c_i to c_j in $G' = G - C + c_i + c_j$.*

Proof. Note that both $[c_i, c_j]$ and $[c_j, c_i]$ have at least d edges. Let P be the shortest path from c_i to c_j in G'. Then both $P + [c_i, c_j]$ and $P + [c_j, c_i]$ are cycles of length at most $|C| - d + d'$ and one of them is odd. Since C is a minimal odd cycle, $|C| \leq |C| - d + d'$, and thus $d \leq d'$. ∎

The next lemmas shows that τ satisfies condition (2).

Lemma 5. *Suppose $C = (c_1, \ldots, c_s)$ is a minimum cycle in a triangle-free graph G. If f is a proper k-coloring of C such that the distance in C between any two vertices of C with the same color is at least four. Then f can be extended to a proper k-coloring f' of $N[C]$.*

Proof. For each vertex $x \in N(C)$, choose a vertex $c_{i(x)}$ such that x is adjacent to $c_{i(x)}$. Let $f'(x) = f(c_{i(x)+1})$, where addition is modulo s. To see that f' is a proper coloring, first consider two adjacent vertices x and y in $N(C)$. Since G is triangle-free, $c_{i(x)} \neq c_{i(y)}$. The distance d' between $c_{i(x)}$ and $c_{i(y)}$ in $G' = G - C + c_{i(x)} + c_{i(y)}$ is at most three. If $f'(x) = f'(y)$, then $f(c_{i(x)+1}) = f(c_{i(y)+1})$ and the distance d between $c_{i(x)}$ and $c_{i(y)}$ in C is at least four. But this contradicts Lemma 4. Next consider $x \in N(C)$ and $c_j \in C$ such that x is adjacent to c_j. If $j = i(x)$ then $f'(x) = f(c_{i(x)+1}) \neq f(c_j) = f'(c_j)$. Otherwise, the distance between $c_{i(x)}$ and c_j in G' is at most two. If $f'(x) = f'(c_j)$, then $f(c_{i(x)+1}) = f(c_j)$ and the distance between $c_{i(x)}$ and c_j is at least three. Again this contradicts Lemma 4 and so Lemma 5 is proved ∎

Finally we show that τ satisfies condition (3).

Lemma 6. *Let $G \in \Gamma$. If a vertex x is adjacent to an odd cycle C in G, then x is the end point of an induced P_{r+1} in $C + x$.*

Proof. Let $c_1, \ldots, c_p$ be the neighbors of x in C. Partition C into paths $P_i = [c_i, c_{i+1})$, $i = 1, \ldots, p$, where addition is modulo p. Since $|C|$ is odd $|P_i|$ is odd for some i. Then $P_i + c_{i+1} + x$ is an odd cycle, and thus has length at least $r + 2$. It follows that $P_i + x$ is an induced path on at least $r + 1$ vertices. ■

We say that two disjoint subgraphs H_1 and H_2 of a graph G are *adjacent* if some vertex v_1 of H_1 is adjacent to some vertex v_2 of H_2. Otherwise we say that H_1 and H_2 are *non-adjacent*. Similarly, we say that H_1 and H_2 are *connected* by a path of length d if there exist a P_{d+1} with one end point in H_1 and the other end point in H_2

The next lemma will be used to show that we need only use a bounded number of global colors.

Lemma 7. *Let $G \in \Gamma$. There exist a function $f(d)$ such that each vertex x of G is connected by a path of length d to at most $f(d)$ pairwise non-adjacent odd cycles in G.*

Proof. We argue by induction on d. For the base step, suppose $d = 1$ and set $f(d) = k - 1$. If a vertex x is adjacent to k pairwise non-adjacent odd cycles, then by Lemma 6, G induces S_k^r which is a contradiction.

Now assume the result holds for $d \leq t$ and consider $d = t + 1$. Set $f(t+1) = (k-1)f^2(t)$. Suppose a vertex x is connected to $f(d)+1$ pairwise non-adjacent odd cycles by paths of length at most d. Choose a minimal set of vertices $F \subset N(x)$ such that $f(d) + 1$ odd cycles are connected to x by paths of length at most d which pass through F. Then by the induction hypothesis, $|F| \geq \lceil (f(d)+1)/f(t) \rceil = (k-1)f(t)+1$. Say $F = \{v_1, \ldots, v_p\}$. For each vertex v_i in F, there exists an odd cycle D_i such that

(*) D_i is connected to x by a path of length at most d which passes through v_i but D_i is not connected to x by a path of length at most d, which passes through any other vertex of F.

For $i = 1, \ldots, q$, let Q_i be the shortest part from D_i to x which passes through v_i. Note that each Q_i is an induced path and, by Lemma 3, can be extended using additional vertices from D_i to an induced path Q_i' on $r + 1$ vertices. We intend to obtain a contradiction by finding an induced S_k^r in $Q_1' + \ldots + Q_p'$. By (*) the paths $R_i = Q_i' - x$ are pairwise disjoint. Say $R_i = (v_i = v_{i,1}, \ldots, v_{i,d-1})$. Also by (*) and the fact that the D_i are pairwise non-adjacent, $v_{i,a}$ is not adjacent to $v_{j,b}$ if $i \neq b$ and $a \neq b$. Since G is triangle-free, $v_{i,1}$ is not adjacent to $v_{j,1}$. Choose $F' \subset \{v_{1,2}, \ldots, v_{p,2}\}$

minimal with respect to the condition that each D_i, $i = 1,, \ldots, p$, can be reached by a path of length at most t from some vertex in F'. Applying the induction hypothesis again, we see that $|F'| \geq \lceil((k-1)f(t)+1)/f(t)\rceil = k$. Without loss of generality, $F' = \{v_{1,2}, \ldots, v_{k,2}\}$. Then $v_{i,a}$ is not adjacent to $v_{j,b}$ if $i \neq j$. But then $Q'_1 + \ldots + Q'_k$ is induced S^r_k, which is a contradiction. ∎

When $d = 2$ the induction step can be simplified to provide the bound $f(d) = (k-1)^2$. This then gives $f(3) = (k-1)^5$, which we will use later.

Proof of Theorem 1. Let $G = (V, E)$ be a graph in Γ. Partition the vertices of G into subsets $V_1, \ldots, V_n$ inductively as follows. Let $G_0 = G$. Suppose we have defined G_i and V_j for all $j < i$. If $\chi(G) \leq 2$, set $n = i$ and let $D_i = V(G_i)$; otherwise let D_i be a minimum odd cycle in G_i. Let $V_i = N[D_i]$, $N_i = N(D_i)$, and $G_{i+1} = G_i - V_i$. Note that for $i \neq j$, D_i is not adjacent to D_j.

We now define a $13[(k-1)^5 + 1]$ coloring ϕ of G. Each vertex x will receive a two coordinate color $\phi(x) = (\phi_\ell(x), \phi_g(x))$. The first coordinate is chosen from eleven local colors; it assures that if two vertices in the same part V_i receive the same color, then they are non-adjacent. The second coordinate is chosen from $(k-1)^5 + 1$ global colors; it assures that if two vertices from distinct parts V_i and V_j receive the same color, then they are non-adjacent. In each case we first color the D_i and then extend the coloring to the N_i. We begin by coloring $G - V_n$.

For $i = 1, \ldots, n-1$, let ϕ_ℓ restricted to D_i be such that the distance between any two vertices which receive the same color is at least seven. This is easily accomplished using 13 colors. Next extend ϕ_ℓ restricted to D_i to V_i as in the proof of Lemma 5. In particular, for each vertex $x \in N_i$, choose a vertex $c_x \in D_i$ such that x is adjacent to c_x. This clearly accomplishes our objective for the local coordinate.

For $i = 1, \ldots, n-1$, we define ϕ_g restricted to D_i by recursion on i. For each vertex $x \in D_i$, let $\phi_g(x)$ be the least color such that (*) if $y \in D_j$, where $j < i$, $\phi_\ell(x) = \phi_g(y)$, and x is connected to y by a path of length at most three, then $\phi_g(x) \neq \phi_g(y)$. Note that if x is connected to two vertices y and z in D_j by paths of length at most three, then the distance d' between y and z in $G_j - D_j + y + z$ is at most six. Thus by Lemma 4 and the choice of ϕ_ℓ at most one vertex y from any D_j can witness the hypothesis of (*). By Lemma 7 at most $(k-1)^5$ of the j's can witness the the hypothesis of (*). Thus ϕ_g restricted to D_i is a $(k-1)^5 + 1$ coloring. Extend ϕ_g to N_i setting $\phi_g(x) = \phi_g(c_x)$. To see that this accomplishes our objective for the global coordinate, consider $x \in V_i$ and $y \in V_j$, with $j < i$. Set $c_x = x$, if $x \in D_i$, and $c_y = y$, if $y \in D_j$. If $\phi_\ell(x) = \phi_\ell(y)$ and x is adjacent to y, then c_x is connected to c_y by a path of length at most three. Thus $\phi_g(x) \neq \phi_g(y)$.

For $i = n$, we can treat V_n in exactly the same way we treated D_j in the previous cases ■

So far complete bipartite graphs, almost complete multipartite graphs, and odd cycles have played the role of templates in successful arguments. The author believes that the template formulation presented here provides a useful framework for attacking Conjecture 1 and that further advances will be made by developing more elaborate templates.

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H. A. Kierstad

Department of Mathematics,
Arizona State University,
Tempe AZ 85287,
U.S.A.