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A Semi-Integral Total Colouring

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ABSTRACT

Behzad and Vizing have conjectured that given any simple graph $G = (V, E)$ of maximum degree Δ, one can colour $V \cup E$ with $\Delta + 2$ colours so that no two adjacent or incident elements are assigned the same colour. We show that there is a $\Delta + 2$ vertex-integral fractional total colouring of $V \cup E$.

1. Preliminaries

A *total colouring* of a graph $G = (V, E)$ is a colouring of $V \cup E$ so that no two adjacent vertices, or two incident edges, or an edge and either of its ends receive the same colour. A long standing conjecture of Behzad [1] and Vizing [8] asserts that any simple graph G of maximum degree Δ admits a $\Delta + 2$ total colouring. Said another way, $V \cup E$ can be partitioned into $\Delta+2$ *total stable sets.* The conjecture has been verified for only very special classes of graphs (see [2] and [7]). In general, the best total colouring known, due to Hind [5], requires at most $\Delta + 2\lceil\sqrt{\Delta}\rceil + 1$ colours. In this paper we formulate the general fractional total colouring problem for simple graphs and present a solution to it which takes a $\Delta + 2$ vertex colouring of G and extends it to a $\Delta + 2$ fractional total colouring. In other words, it is shown that when fractions of total stable sets are allowed, there is a solution to the

Behzad-Vizing conjecture whose restriction on V is a $\Delta+2$ vertex colouring of G.

Let $\mathcal{T}$ denote the family of all total stable sets of G and let $\bar{1} = (1,\cdots,1)$. A *fractional total colouring* of a graph G is a point w in

$$P = \left\{ w : w \geq 0; \sum_{T \ni u} w_T \geq 1, u \in E \cup V, T \in \mathcal{T} \right\}.$$

If in addition there exists a family $\mathcal{B}$ of disjoint stable sets of G such that, $|\mathcal{B}| \leq \bar{1}w$ and for every $T \in \mathcal{T}$ with $w_T > 0$, $V \cap T \in \mathcal{B}$ then w is a *vertex-integral* fractional total colouring of G. In a recent paper [6] the authors have shown that $\min\{\bar{1}w : w \in P\} \leq \Delta+2$ for any simple graph G. It is our intention here to strengthen this result. By combining similar techniques and exploring different properties of the fractional edge colouring problem (to be defined below), we show that the above inequality holds true even when we require that w is a vertex-integral fractional total colouring of G. Regarding the range of values of $\min\{\bar{1}w : w \in P\}$ we note that it is bounded below by $\Delta + 1$ and that it is easy to construct simple graphs for which it is exactly $\Delta + 2$. We prove the main theorem in section 2. In the remainder of this section we review some results, which we will need in our proof, concerning fractional edge colourings of graphs.

For any graph G, consider the polyhedron

$$P' = \left\{ y : y \geq 0; \sum_{M \ni e} y_M \geq c_e, e \in E, M \in \mathcal{M} \right\}$$

and the linear program

$$\min\{\bar{1}y : y \in P'\}, \tag{1}$$

where $c_e \in Q^+$ and $\mathcal{M}$ is the family of all matchings of G. A point $y \in P'$ is called a *weighted fractional edge colouring* of G. If $c = \bar{1}$, then y is a *fractional edge colouring* of G. The dual of (1) is

$$\max\{cx : x \in Q_M\},$$

where

$$Q_M = \left\{ x : x \in \mathbf{R}_+^{|E|};\ x(M) \leq 1, M \in \mathcal{M} \right\}.$$

The theory of antiblocking polyhedra developed by Fulkerson [4], tells us that the antiblocker $A(Q_M)$ of Q_M is the convex hull of the incidence vectors of matchings of G. Edmonds [3] has shown that

$$\begin{aligned} A(Q_M) = \{x : x \in \mathbf{R}_+^{|E|};\ & x(\delta(v)) \leq 1, v \in V; \\ & x(E(U)) \leq \lfloor |U|/2 \rfloor, U \subseteq V\}, \end{aligned}$$

where $E(U) = \{uv \in E : \{u, v\} \subseteq U\}$. Hence,

$$\max\{cx : x \in Q_M\} = \max_{\substack{v \in V(G) \\ U \subseteq V(G)}} \left\{ \sum_{e \in \delta(v)} c_e, \Big(\sum_{e \in \gamma(U)} c_e \Big) / \lfloor |U|/2 \rfloor \right\}. \quad (2)$$

The last equality suggests the following definition. An induced subgraph H of G is called *overfull* if $|E(H)| > \Delta(|V(H)| - 1)/2$. Note that if H is overfull, then $|\delta(H)| \leq \Delta - 1$ and when G is simple $|V(H)| \geq \Delta + 1$. Also, it is easy to check that for $c = \bar{1}^T$, the maximum in (2) is Δ, unless G contains an overfull subgraph with an odd number of vertices in which case it is more than Δ. As a consequence, if G has no such subgraphs, then it has a fractional edge colouring y with $\bar{1}y = \Delta$. An overfull subgraph of G is *minimal* if no proper subset of its vertices induces an overfull subgraph of G. Overfull subgraphs, particularly minimal ones, turn out to be pivotal in our treatment of the fractional total colouring problem.

2. The colouring

Theorem. *Any simple graph G has a $\Delta + 2$ vertex-integral fractional total colouring.*

Proof. The main idea of the proof is easy to explain. A $\Delta + 2$ vertex colouring of G is first selected. Then, for each colour class S_i, $1 \leq i \leq \Delta + 2$, a weighted fractional edge colouring of $G - S_i$ is obtained by solving (1). Finally, these two colourings of vertices and edges of G are suitably combined to obtain the desired fractional total colouring.

It is shown in [2] and [7] respectively that the theorem holds for complete graphs and simple graphs of maximum degree three. So we assume, without loss of generality, that no component of G is a complete subgraph on $\Delta + 1$

vertices and that $\Delta \geq 4$. Furthermore, in order to simplify the argument used, and the presentation of the proof, we will take G to be Δ-regular. Observe that if it is not, then one can create a sequence of simple graphs $G^0, \cdots, G^l$, where $G^0 = G$, G^l is Δ-regular and G^i is obtained by duplicating G^{i-1} and joining each vertex of degree less than Δ to its image. Of course, if there is a colouring of G^l which satisfies the statement of the theorem, then so does its restriction to G.

Clearly, if G is Δ-regular then it has a $\Delta + 2$ vertex colouring (just colour $V(G)$ greedily). In fact, if we can choose a $\Delta + 2$ vertex colouring of G such that for each colour class S_i, $G - S_i$ contains no overfull subgraph, then it is easy to give a vertex-integral fractional total colouring of G. For each i, we will obtain a Δ fractional edge colouring of $G - S_i$ and then combine the matchings in this colouring with S_i to obtain the total stable sets containing S_i. The details are explained below.

Consider a $\Delta + 2$ vertex colouring of G and suppose further that for each colour class S_i, $1 \leq i \leq \Delta + 2$, $G - S_i$ contains no overfull subgraph of G and thus has a fractional edge colouring y^i satisfying $\bar{1}y^i = \Delta$. Let $\mathcal{M}_i$ denote the family of all matchings of $G - S_i$. We can now obtain a fractional total colouring that meets the statement of the theorem. For each i, $1 \leq i \leq \Delta + 2$, let $T_{ij} = S_i \cup M_{ij}$ and $w_{T_{ij}} = y^i_{M_{ij}}/\Delta$ for all $M_{ij} \in \mathcal{M}_i$. For all other total stable sets T of G, let $w_T = 0$. Clearly, each T_{ij} is a total stable set of G. Also, for each node $v \in S_i$ we have that

$$\sum_{T \ni v} w_T = \sum_{M_{ij} \in \mathcal{M}_i} (y^i_{M_{ij}}/\Delta) = \Delta/\Delta = 1,$$

and for each edge $e \in E(G)$, with one end in S_l and one end in S_m,

$$\sum_{T \ni e} w_T = \sum_{\substack{i=1 \\ i \notin \{l\ m\}}}^{\Delta+2} \sum_{\substack{M_{ij} \in \mathcal{M}_i: \\ M_{ij} \ni e}} (y^i_{M_{ij}}/\Delta) \geq \sum_{\substack{i=1 \\ i \notin \{l\ m\}}}^{\Delta+2} (1/\Delta) = \Delta/\Delta = 1.$$

Moreover,

$$\bar{1}w = \sum_{i=1}^{\Delta+2} \sum_{M_{ij} \in \mathcal{M}_i} (y^i_{M_{ij}}/\Delta) = \sum_{i=1}^{\Delta+2} 1 = \Delta + 2$$

as required.

Motivated by the discussion above, we will now consider for which graphs we can obtain a $\Delta+2$ vertex colouring such that $G - S_i$, $1 \leq i \leq \Delta+2$, contains no overfull subgraphs of G. Or equivalently, we will consider graphs whose vertex set can be partitioned into colour classes $S_1, \cdots, S_{\Delta+2}$ such that, $S_i \cap V(H) \neq \emptyset$ for each colour class S_i and each overfull subgraph H of G. It turns out that these are exactly the graphs that contain no overfull subgraph of size $\Delta + 1$. We prove this in a moment. We will deal with graphs which contain overfull subgraphs on $\Delta + 1$ vertices later. The proof in both cases relies on the fact that the minimal overfull subgraphs of G are disjoint, a fact we prove now.

We will need the following notation. For any two disjoint subgraphs F_1, F_2 of G, $E(F_1, F_2)$ denotes the set of edges of G with one end in F_1 and the other in F_2.

Lemma 1. *Let F_1, F_2 be overfull subgraphs of a graph G. If $|E(F_1 \cap F_2, F_1 \backslash F_2)| \leq |E(F_1 \cap F_2, F_2 \backslash F_1)|$, then $F_1 \backslash F_2$ is overfull.*

Proof. We have that $|E(F_1 \cap F_2)| + |E(F_1 \cap F_2, F_1 \backslash F_2)|$
$\leq (|V(F_1 \cap F_2)|\Delta - |E(F_1 \cap F_2, G \backslash (F_1 \cap F_2))|)/2 + |E(F_1 \cap F_2, F_1 \backslash F_2)|$
$\leq |V(F_1 \cap F_2)|\Delta/2$.
Thus, $|E(F_1 \backslash F_2)| = |E(F_1)| - |E(F_1 \cap F_2, F_1 \backslash F_2)| - |E(F_1 \cap F_2)|$
$> (|V(F_1)| - 1)\Delta/2 - |V(F_1 \cap F_2)|\Delta/2 = (|V(F_1 \backslash F_2)| - 1)\Delta/2$.
Therefore, $F_1 \backslash F_2$ is overfull. ■

Corollary 2. *Any two minimal overfull subgraphs of a graph G are disjoint.*

Corollary 2 can be used to show that if all overfull subgraphs of G contain at least $\Delta + 2$ vertices, then there exists a $\Delta + 2$ vertex colouring of G such that each overfull subgraph of G contains a vertex of each colour. To this end, consider a graph G with no overfull subgraph of size $\Delta + 1$ and let H be a minimal overfull subgraph of G (if G has no such subgraph, any $\Delta + 2$ colouring of V suffices). If H is the only minimal overfull subgraph of G, then we can choose a set D of $\Delta + 2$ vertices in H, give them all distinct colours and then extend this greedily to a colouring of V. If there is another minimal overfull subgraph in G, then, by induction, we can colour $V(G \backslash H)$ such that every overfull subgraph of $G \backslash H$ contains a vertex of each colour. By corollary 2, to obtain our desired colouring, we need only extend this to a colouring of G in which every colour appears in H. Again, we will choose a subset $D = \{v_1, \cdots, v_{\Delta+2}\}$ of $V(H)$, give these vertices distinct colours, and then extend greedily on $V(H) - D$ for a complete colouring of V. Let $\delta(H)$ denote the set of edges with exactly one end in $V(H)$. We choose D so that it contains all the vertices of H which are ends of edges of $\delta(H)$ and such that, if v_i is the end of some edge of $\delta(H)$ and $i > j$ then v_j is the end of some edge of $\delta(H)$. Having coloured $\{v_1, \cdots, v_{i-1}\}$, we choose a colour for v_i different from those given to $\{v_1, \cdots, v_{i-1}\}$ as follows. If there is an edge $v_i w$ with $w \in V(G \backslash H)$, then by our choice of D, $|\{w : w \in V(G\backslash H), v_i w \in E\}| \leq |\delta(H)| - (i-1) \leq \Delta - i$ and hence, there are at least three choices for the colour of v_i. If there is no such an edge, then we simply colour v_i with any colour not used by $\{v_1, \cdots, v_{i-1}\}$. It follows that we can obtain our desired colouring of G.

From the above remarks, we can restrict our attention to graphs which have at least one overfull subgraph of size $\Delta + 1$. Hence, $G = G' \bigcup_{k=1}^{s} H_k$, where the H_k's are disjoint overfull subgraphs of G each of order $\Delta + 1$ and G' contains no such subgraph. Furthermore, the method of the previous paragraph, with the obvious modifications, can be used to obtain a partitioning of the vertices of G into colour classes $S_1, \cdots, S_{\Delta+2}$ such that, for every overfull subgraph H of G, $S_i \cap V(H) \neq \emptyset$, $1 \leq i \leq \Delta + 1$, and $S_{\Delta+2} \cap V(H) \neq \emptyset$ when $|V(H)| \geq \Delta + 2$.

We proceed with describing a weighted fractional edge colouring in $G - S_i$, $1 \leq i \leq \Delta + 2$. First, for each k, $1 \leq k \leq s$, we select a set A_k of $\lceil (\Delta+2)/2 \rceil$ vertices such that $A_k \subseteq \{v : v \in H_k, |N(v) \cap V(H_k)| \geq \Delta - 1\}$ and $|N(v) \cap V(H_k)| \geq |N(u) \cap V(H_k)|$ for any $u \in V(H_k) - A_k$, $v \in A_k$. Observe that if t is the cardinality of $\{v : v \in H_k, |N(v) \cap V(H_k)| \geq \Delta - 1\}$, then $(t\Delta + (\Delta + 1 - t)(\Delta - 2))/2 \geq |E(H_k)| > \Delta^2/2$. This implies that $t > (\Delta + 2)/2$ and therefore such an A_k exists for each k. Let $K_i = \{k : S_i \cap A_k \neq \emptyset, 1 \leq k \leq s\}$ and for each $k \in K_i$ let v_{ik} be the vertex in $S_i \cap A_k$. Next, for each colour class S_i we define a vector c^i on the edges of $G - S_i$ as follows. For each i, $1 \leq i \leq \Delta + 1$, we let $c_e^i = 1 + \alpha / |E(G[N(v_{ik}) \cap V(H_k)])|$ if $e \in E(G[N(v_{ik}) \cap V(H_k)])$ for some $k \in K_i$, and $c_e^i = 1$ otherwise, where $\alpha = \Delta / 2\lceil (\Delta + 2)/2 \rceil$. (An expression for $|E(G[N(v_{ik}) \cap V(H_k)])|$ will be given below. We note for now that it is not equal to zero). Similarly, we obtain $c^{\Delta+2}$ by setting $c_e^{\Delta+2} = \Delta - \sum_{i=1, i \notin \{l,m\}}^{\Delta+1} c_e^i$, $e \in E(G - S_{\Delta+2})$, where l, m index the colour classes in which e has an end. Note that $c_e^{\Delta+2} = 1 - \theta_e$, where $\theta_e = (\sum_{i=1, i \notin \{l,m\}}^{\Delta+1} c_e^i) - (\Delta - 1)$. Finally, with each vector c^i and subgraph $G - S_i$, $1 \leq i \leq \Delta + 2$, we obtain a fractional weighted edge colouring y^i by solving (1).

We first show that if for each i, $1 \leq i \leq \Delta + 2$, $\bar{1} y^i$ is at most Δ, then we can obtain a $\Delta + 2$ vertex-integral fractional total coulouring of G. By increasing some components of each y^i, if necessary, we shall assume that $\bar{1}y^i = \Delta$. We can also ensure that in $G - S_{\Delta+2}$, whenever $c_e^{\Delta+2} < 1$ we have that $\sum_{M \ni e} y_M^{\Delta+2} = c_e^{\Delta+2}$ by dropping e from some total stable sets. For each i, $1 \leq 1 \leq \Delta + 2$, let $T_{ij} = S_i \cup M_{ij}$ and $w_{T_{ij}} = y^i_{M_{ij}}/\Delta$ for all $M_{ij} \in \mathcal{M}_i$. For any other total stable set T of G, let $w_T = 0$. Clearly each T_{ij} is a total stable set of G. Moreover, w is a fractional total colouring of G. Indeed, for any $v \in S_i$, $1 \leq i \leq \Delta + 2$,

$$\sum_{T \ni v} w_T = \sum_{M_{ij} \in \mathcal{M}_i} (y^i_{M_{ij}}/\Delta) = \Delta/\Delta = 1,$$

and for any $e \in E(G)$, with one end in S_l and the other in S_m,

$$\sum_{T \ni e} w_T = \sum_{\substack{i=1 \\ i \notin \{l\ m\}}}^{\Delta+2} \sum_{\substack{M_{ij} \in \mathcal{M}_i: \\ M_{ij} \ni e}} (y^i_{M_{ij}}/\Delta) \geq (\Delta + \theta_e - \theta_e)/\Delta = 1.$$

Finally,

$$\bar{1}w = \sum_{i=1}^{\Delta+2} \sum_{M_{ij} \in \mathcal{M}_i} (y^i_{M_{ij}}/\Delta) = \sum_{i=1}^{\Delta+2} 1 = \Delta + 2$$

and as the solution presented requires at most $\Delta + 2$ stable sets, the proof of our claim is complete.

It remains to show that for each y^i, $1 \leq i \leq \Delta + 2$, $\bar{1}y^i$ is at most Δ and that $c_e^i \in Q^+$ for each $e \in E(G - S_i)$. We will make use of the following lemma.

Lemma 3. *Let F be an overfull subgraph of a Δ-regular simple graph G with $|V(F)| = \Delta + 1$ and let F' be an induced subgraph of F with $2 \leq |V(F')| \leq \Delta - 1$. Then $|\delta(F')| \geq 2(\Delta - 1)$ and $|E(F', F \backslash F')| \geq 3(\Delta - 1)/2$.*

Proof. Observe that $|\delta(F')| = \Delta|V(F')| - 2|E(F')| \geq \Delta|V(F')| - |V(F')|(|V(F')| - 1) = (\Delta + 1 - |V(F')|)|V(F')|$. Hence for $|V(F')| = 2$ (or $|V(F')| = \Delta - 1$), $|\delta(F')|$ is minimized and the first statement of the lemma follows. The second claim can be as easily derived from the fact that F is overfull: $|E(F', F)| \geq |E(F)| - |E(F')| - |E(F \backslash F')| \geq (\Delta^2 + 1)/2 - (|E(F')| + |E(F \backslash F')|) \geq (1 - \Delta)/2 + |V(F)|(\Delta + 1 - |V(F)| \geq (3\Delta - 1)/2$ when $2 \leq |V(F')| \leq \Delta - 1$.

We show that $\bar{1}y^i \leq \Delta$, $1 \leq i \leq \Delta+2$, by establishing that the maximum in (2) is at most Δ for each c^i and $G - S_i$. To begin, we prove that $c^i_e \geq 0$, $e \in E(G - S_i)$. This is certainly true when $i \in \{1, \cdots, \Delta + 1\}$. So let $i = \Delta + 2$ and let H be an overfull subgraph of G of size $\Delta + 1$. Note that if $v \in V(H)$ and $|N(v) \cap V(H)| \geq \Delta - 1$, then $|E([N(v) \cap V(H)])| \geq (\Delta - 2)^2/2$. Thus, $c^{\Delta+2}_e = 1 - \theta_e \geq 1 - \lceil(\Delta + 2)/2\rceil 2\alpha/(\Delta - 2)^2 = 1 - \Delta/(\Delta - 2)^2 \geq 0$ when $\Delta \geq 4$, as required. Next, observe that in $G - S_i$, $1 \leq i \leq \Delta + 2$, corollary 2 and the choice of c^i indicate that $\sum_{e \in \delta(v)} c^i_e \leq \Delta$, $v \in V$. Hence, according to (2), we need only be concerned with the edge weight of certain induced subgraphs of $G - S_i$. In other words, we have to show that $\sum_{e \in E(H)} c^i_e \leq \Delta \lfloor |V(H)|/2 \rfloor$, for each induced subgraph H of $G - S_i$. The fact that $\sum_{e \in \delta(v)} c^i_e \leq \Delta$ ($v \in V$), easily implies that the inequality is true when H has an even number of vertices. Thus we need only concern with those H that have an odd number of vertices.

Suppose that $i \in \{1, \cdots, \Delta + 1\}$. Let H be an induced subgraph of $G - S_i$ with an odd number of vertices. There are two possibilities:

1) for all $e \in E(H)$ we have that $c^i_e = 1$. This is the easy case. By construction $G - S_i$ does not contain an overfull subgraph, thus $\sum_{e \in E(H)} c^i_e = |E(H)| \leq \Delta(|V(H)| - 1)/2$.
2) there is an $e \in E(H)$ such that $c^i_e > 1$. Let t be the number of indices $k \in K_i$ such that there is an $e \in E(H) \cap E(G[N(v_{ik}) \cap V(H_k)])$. We know that $\sum_{e \in E(H)} c^i_e \leq |E(H)| + t\alpha$. Hence, it suffices to show that $|E(H)| \leq \Delta \lfloor |V(H)|/2 \rfloor - t\alpha$. It is therefore enough to prove that in G,

$$|\delta(H)| \geq \Delta + 2t\alpha. \tag{3}$$

Consider the subcases:

i.) $t \geq 2$. Define $V(F'_k) := V(H) \cap V(H_k)$. If $2 \leq |V(F'_k| \leq \Delta - 1$ then we can apply lemma 3 to conclude that $|E(F'_k, H_k \backslash F'_k)| \geq 3(\Delta - 1)/2$. If $|V(F'_k)| = \Delta$ then $|E(F'_k, H_k \backslash F'_k)| \geq \Delta - 1$ and $|E(F'_k, H_k \backslash F'_k)| \geq \Delta$ when $\Delta = 4$. This is because $v_{ik} \in A_k$ and because of the definition of A_k. It follows that $|\delta(H)| \geq t(\Delta - 1)$ and $|\delta(H)| \geq t\Delta$ when $\Delta = 4$. Hence with $t \geq 2$, (3) is satisfied.

ii.) $t = 1$. We shall show that $|\delta(H)| \geq \Delta + 2 \geq \Delta + 2\alpha$. Since $|V(H)|$ is odd and G is Δ-regular, $|\delta(H)|$ and Δ have the same parity. Thus, we need only show that $|\delta(H)| \geq \Delta + 1$. Let H_k be the overfull subgraph of G of order $\Delta+1$ for which $E(H) \cap E(H_k) \neq \emptyset$. Let $H' = H \backslash H_k$ and suppose for the moment that $H' = \emptyset$. If $H \neq H_k - v_{ik}$, then lemma 3 immediately implies (3). Thus we can assume that $H = H_k - v_{ik}$. Since G is Δ-regular with no component being a complete subgraph on $\Delta+1$ vertices and $|V(H_k)| = \Delta+1$, we have that $|\delta(H_k)| \geq 2$. Now if $|\delta(H_k)| = 2$, then $|N(v_{ik}) \cap V(H_k)| = \Delta$ and hence $|\delta(H)| = \Delta + 2$. Otherwise, $|\delta(H_k)| \geq 4$ and again, by the way A_k was selected, $|\delta(H)| \geq (\Delta-1)+3 = \Delta+2$. Therefore, $H' = \emptyset$ implies that (3) holds. It remains to examine the case in which $H' \neq \emptyset$. First observe that, by hypothesis, $|\delta(H')| \geq \Delta$ and $\delta(H_k) \leq \Delta - 1$. Suppose further that $H_k \backslash H = \{v_{ik}\}$ and note that $|E(H', H_k - v_{ik})| \leq \Delta - 1$ when $N(v_{ik}) \subseteq V(H_k)$ and $|E(H', H_k - v_{ik})| \leq \Delta - 2$ otherwise. This easily implies that $|\delta(H)| \geq \Delta + 1$. Finally, when $H_k \backslash H \neq \{v_{ik}\}$, lemma 3 and the fact that $|\delta(H')| \geq \Delta$ imply (3). We conclude that for any t, (3) holds and therefore $\bar{1}y^i \leq \Delta$, $1 \leq i \leq \Delta+1$, as required.

It remains to consider the weighted fractional edge colouring $y^{\Delta+2}$. Note that due to the edge weights used, if the minimum in (1) is greater than Δ, then the only candidate subgraphs of $G - S_{\Delta+2}$ to maximize (2) will be overfull subgraphs of G. Moreover, the way G was vertex coloured ensures that any such subgraph has to contain an overfull subgraph H_k of G on $\Delta + 1$ vertices. But then, $\sum_{e \in E(H)} c_e^{\Delta+2} \leq |E(H)| - |A_k|\alpha \leq \Delta|V(H)|/2 - \lceil(\Delta+2)/2\rceil\alpha \leq \Delta\lfloor|V(H)|/2\rfloor$. Therefore, $\bar{1}y^{\Delta+2} \leq \Delta$. This completes the proof of the theorem.

A few concluding remarks are in order. In a straight forward manner, edge integrality can replace vertex integrality in the definition of a vertex-integral fractional total colouring. (Both these definitions and the associated problems have been suggested to us by U.S.R. Murty). In this context, a k total colouring of a graph G can be viewed as being a k vertex- and edge-integral fractional total colouring of G. And conversely, it can be shown that given a k vertex- and edge-integral fractional total colouring one can obtain a k total colouring of G. It would therefore be of interest to examine the Behzad-Vizing conjecture in light of edge-integral fractional total colourings.

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