COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
60. SETS, GRAPHS AND NUMBERS, BUDAPEST (HUNGARY), 1991

The Chromatic Number of some Uncountable Graphs

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ABSTRACT

Construct a graph on the set of all $\omega_2 \to \omega$ functions by joining two of them if they are eventually different. P. Erdős and A. Hajnal posed the problem of determining the chromatic number of this graph. They proved, that, under CH, it is at least $\aleph_1$, and it is consistent that it is at least $\aleph_2$. We show that (under GCH) both values $\aleph_2$ and $\aleph_3$ can be attained.

If $\kappa \geq \omega$ is a regular cardinal, let $G(\kappa)$ be the following graph. The vertex set is the collection of all $\kappa \to \omega$ functions. Two such functions are joined iff they differ from some point onward. The graph $G(\omega_2)$ was introduced and studied by P. Erdős and A. Hajnal in [3]. They investigated the problem if there exist uncountably chromatic graphs of size $\aleph_2$ such that every subgraph of size $\aleph_1$ is countably chromatic. They gave an example of such a graph under the continuum hypothesis and showed that (even without CH) every such graph can be embedded into $G(\omega_2)$. This implies, under CH, that $\mathrm{Chr}(G(\omega_2)) \geq \aleph_1$ (Chr means chromatic number). They also proved that $\mathrm{Chr}(G(\omega_2)) \geq \aleph_2$ if GCH holds and there exists an ω_2-Kurepa tree.

In [3] it is also asked if there exists a graph of size and chromatic number $\aleph_2$ in which every subgraph of size $\aleph_1$ is ω-chromatic. Baumgartner [2] proved that the existence of such a graph is consistent with GCH.

Foreman and Laver [4] showed the independence of the statement, from the consistency of the existence of a huge cardinal. Recently, Shelah [6] produced graphs of this kind under the axiom of constructibility.

We start with an easy observation.

Theorem 1. *If* $2^\omega = 2^{\omega_1} < 2^{\omega_2}$ *then* $\mathrm{Chr}(G(\omega_2)) \geq \omega_2$.

Proof. By the cardinal arithmetic hypothesis there exist functions $f_\alpha : \omega_2 \to 2^\omega$ for $\alpha < 2^{\omega_2}$ such that if $\alpha \neq \beta$ then $f_\alpha(\xi) \neq f_\beta(\xi)$ for ξ large enough. Let $\{A(\alpha) : \alpha < 2^\omega\}$ be a family of independent infinite subsets of ω. If $\alpha < \beta < 2^{\omega_2}$, put $F_{\alpha\beta}(\xi) = \min(A(f_\alpha(\xi)) - A(f_\beta(\xi)))$ for $\xi < \omega_2$. Clearly, $F_{\alpha\beta} : \omega_2 \to \omega$. If we color the functions $\{F_{\alpha\beta} : \alpha < \beta < 2^{\omega_2}\}$ by $\aleph_1$ colors, then by the Erdős–Rado theorem, as $2^{\aleph_2} > 2^{\aleph_1}$, for some $\alpha < \beta < \gamma$, $F_{\alpha\beta}$ and $F_{\beta\gamma}$ get the same color. If $\xi < \omega_2$ is large enough, $f_\alpha(\xi), f_\beta(\xi), f_\gamma(\xi)$ are different, and so $F_{\alpha\beta}(\xi) \neq F_{\beta\gamma}(\xi)$, so they are joined, a contradiction to the fact that they get the same color. ∎

In ZFC, we cannot even settle the following problems.

Conjecture 1. *It is consistent that* $\mathrm{Chr}(G(\omega_2)) = \omega$.

Conjecture 2. *CH is consistent with* $\mathrm{Chr}(G(\omega_2)) = \omega_1$.

Conjecture 1 implies the following conjecture of A. Dow and the author.

Conjecture 3. *It is consistent that every graph G of size $\aleph_2$ with* $\mathrm{Chr}(G) \geq \omega_1$ *has a subgraph with size and chromatic number* $\aleph_1$.

If there is a model for this statement, then, by the above mentioned result of Erdős and Hajnal, $2^\omega \geq \omega_2$ must hold in it.

Theorem 2. *If* $2^\omega \leq \omega_2$ *and there is a nonreflecting stationary subset* $S \subset \omega_2$ *then there is a graph G on ω_2 with* $\mathrm{Chr}(G) = \omega_1$, $\mathrm{Chr}(G') \leq \omega$ *for every subgraph G' of size* $\leq \omega_1$.

Proof. Let $\{A_\alpha : \alpha < \omega_2\}$ be disjoint sets of size 2^ω, $A = \cup\{A_\alpha : \alpha < \omega_2\}$. We build G on A. Put $B_\alpha = \cup\{A_\beta : \beta < \alpha\}$. A point $x \in A_\alpha$ is joined into points in B_α only if $\alpha \in S$. Notice that then $cf(\alpha) = \omega$. Then, if $y \in A_\alpha$, y is joined into a sequence x_n such that $x_n \in A_{\alpha_n}$ where $\alpha_n \nearrow \alpha$. Also, we require, that for every such sequence x_n a y be found. If $f : A \to \omega$ is a good coloring, as S is stationary, there is an $\alpha \in S$ such that for every $i < \omega$ either $f^{-1}(i) \subseteq B_\alpha$ or else there are arbitrarily high $\beta < \alpha$ with $f^{-1}(i) \cap A_\beta \neq \emptyset$. By the construction of our graph, there is a point $y \in A_\alpha$

joined into a point with color i for every $i < \omega$ for which the second clause holds, so $f(y)$ cannot be of either type, a contradiction.

To prove that every subset of A of size $\leq \omega_1$ can be ω-colored, it suffices to show that B_α is ω-chromatic for every $\alpha < \omega_2$. We show this by induction on α. The cases α is a successor, or $cf(\alpha) = \omega$ are trivial. If $cf(\alpha) = \omega_1$, as S is nonreflecting, there is a closed unbounded subset $C \subset \alpha$, disjoint from S. Let $C = \{\gamma_\xi : \xi < \omega_1\}$ be the increasing enumeration of C with $\gamma_0 = 0$. By the inductive hypothesis, all the subsets $D_\xi = \cup\{A_\beta : \gamma_\xi \leq \beta < \gamma_{\xi+1}\}$ are $\leq \omega$-chromatic, $B_\alpha = \cup\{D_\xi : \xi < \omega_1\}$, and, as $C \cap S = \emptyset$, every point in D_ξ is joined into finitely many points in $\cup\{D_\tau : \tau < \xi\}$, therefore giving the claim. ∎

We have not been able to show the consistency of $\mathrm{Chr}(G(\omega_2)) = \omega_1$. Notice that, by the above mentioned result of Erdős-Hajnal, it implies the quoted result of Foreman-Laver. We know the consistency of two other values of $\mathrm{Chr}(G(\omega_2))$ with GCH.

Theorem 3. *If the existence of a huge cardinal is consistent, then so are GCH and* $\mathrm{Chr}(G(\omega_2)) = \omega_2$.

Proof. Magidor [5] gave a model in which there is a uniform ultrafilter D on ω_2 such that $|{}^{\omega_2}\omega/D| = \omega_2$. This obviously implies that the graph in question is $\leq \omega_2$-chromatic. Magidor starts with a huge cardinal κ, a huge embedding $j\colon V \to M$, $j(\kappa) = \lambda$ measurable. He forces by a λ-c.c. partial order, making $\kappa = \omega_1$, $\lambda = \omega_3$. This makes the original $({}^{<\lambda}2, {}^{\lambda}2)$ tree an ω_3-Kurepa tree. Then he collapses ω_1 onto ω, which transforms the latter tree into an ω_2-Kurepa tree. By an argument of Erdős-Hajnal [3], this implies $\mathrm{Chr}(G(\omega_2)) \geq \omega_2$. ∎

Theorem 4. *It is consistent that GCH and* $\mathrm{Chr}(G(\omega_2)) = \omega_3$ *hold.*

Proof. Using a construction of S. Shelah [6], we show that, there consistently exists an ω_3-chromatic graph on ω_3, which can be embedded, by a forcing, into $G(\omega_2)$. We start with the assumption V=L.

A quadruple $p = (s, f, \theta, F)$ is called a *pre-condition* if $s \in [\omega_3]^{\leq \aleph_1}$, f is a function, $\mathrm{Dom}(f) \subseteq [s]^2$, $\mathrm{Ran}(f) \subseteq \omega_2$, $\theta < \omega_2$, $F\colon s \times \theta \to \omega$, and if $(x, y) \in \mathrm{Dom}(f), f(x, y) < \alpha < \theta$ then $F(x, \alpha) \neq F(y, \alpha)$. We use the notation $\theta = \theta^p$, etc. $p' = (s', f', \theta', F') \leq p = (s, f, \theta, F)$ iff $s' \supseteq s, f = f'|[s]^2, \theta' \geq \theta, F' \supseteq F$. Fix an enumeration $\{p(\xi)\colon \xi < \omega_3\}$ of the pre-conditions. For $\varepsilon < \omega_3$ put $p|\varepsilon = (s \cap \varepsilon, f|[s \cap \varepsilon]^2, \theta, F|(s \cap \varepsilon) \times \theta)$ and $\tau(p) = \min\{\varepsilon\colon p|\varepsilon = p\}$.

Whenever X is a graph on ω_3, a pre-condition $p = (s, f, \theta, F)$ is a *condition,* if $\mathrm{Dom}(f) = [s]^2 \cap X$. Let P be the set of all conditions. We first notice that if p, q are conditions, $\theta^p = \theta^q$, then p, q are compatible if and only if f^p, f^q as well as F^p, F^q are compatible as functions. In this case we are going to define the *direct extension* of p, q as $r = (s^p \cup s^q, f^p \cup f^q \cup f, \theta + 1, F^p \cup F^q)$ where $f(x, y) = \theta$ for $x \in s^p - s^q$, $y \in s^q - s^p$, $\{x, y\} \in X$. It is easy to see that $(P, \leq)$ is $< \omega_2$-closed.

Lemma 1. *$(P, \leq)$ is ω_3-c.c.*

Proof. By Δ-system arguments and the above remarks on direct extensions. ■

Lemma 2. *If every subgraph of X of size $\aleph_1$ is countably chromatic, then for every $\eta < \omega_2$, the set $D = \{(s, f, \theta, F): \theta \geq \eta\}$ is dense.*

Proof. If $p = (s, f, \theta, F) \in P, \theta < \eta, g: s \to \omega$ is a good coloring of X on s, put $q = (s, f, \eta, F')$ where $F' \supseteq F$ has $F'(x, \alpha) = g(x)$ for $\theta < \alpha < \eta$. Clearly, $q \leq p$. ■

In the following construction of the graph X we follow some ideas of [6]. Fix a bijection $G: \omega_2 \to \omega_2^3$, and a surjection $e_{\alpha,\beta}: \omega_2 \to [\alpha, \beta)$ for every $\alpha < \beta < \omega_3$. As we assume V=L, the so-called diamond-in-the-square principle holds, see [1]. Therefore, for every limit δ, with $\omega_2 < \delta < \omega_3$, there are a closed, unbounded $C_\delta \subseteq \delta$, and a model $M_\delta = (\delta; K_\delta, \nu_\delta, i_\delta, t_\delta, c_\delta)$ such that $K_\delta: \omega_2 \times \delta^2 \to \delta$, $\nu_\delta: \delta \to \delta$, $i_\delta: \delta \to \omega_2$, $t_\delta: \delta \to \delta$, $c_\delta < \delta$. For $\alpha \in C'_\delta, C_\alpha = \alpha \cap C_\delta$ and $M_\alpha \prec M_\delta$. Whenever $M = (\omega_3; K, \nu, i, t, c)$ is a model with $K: \omega_2 \times \omega_3^2 \to \omega_3$, $\nu: \omega_3 \to \omega_3$, $i: \omega_3 \to \omega_2$, $t: \omega_3 \to \omega_3$, $c < \omega_3$, then for stationary many $\delta \in B = \{\alpha < \omega_3: \mathrm{otp}(C_\alpha) = \omega_2\}, M_\delta \prec M$. Put $h(\delta) = \min(C'_\delta)$. Clearly, $c_\delta < h(\delta)$.

We are going to construct partial functions $g_\delta: C'_\delta \to \delta$ for every limit $\omega_2 < \delta < \omega_3$ by transfinite recursion on δ. $g_\alpha = g_\delta | \alpha$ will hold for $\alpha \in C''_\delta$. Assume that the order type of C'_δ is $\xi + 1$, the ξ-th element of it is ε, we find $g_\delta(\varepsilon)$ as follows. Put $G(\xi) = (i, \mu, \nu)$ for some $i, \mu, \nu < \omega_2$. Let the μ-th interval of C''_δ be $[\alpha, \beta)$, and $\zeta = e_{\alpha,\beta}(\nu)$. If there exists a γ with $\varepsilon < h(\gamma) < \delta$, $\theta^{p(\nu(\gamma))} = \theta^{p(\zeta)}$, such that the conditions $p(\nu(\gamma)), p(\zeta)$ are compatible, $i_\delta(\gamma) = i$, then we let $g_\delta(\varepsilon)$ be the least such γ, otherwise, we leave $g_\delta(\varepsilon)$ undefined. To construct X, we join δ into $\mathrm{Ran}(g_\delta)$ for every $\delta \in B$.

Lemma 3. *In V^P, $\mathrm{Chr}(X) = \omega_3$.*

Proof. Assume that $1 \Vdash H\colon \omega_3 \to \omega_2$ is a good coloring. For every $\delta \in B$ select a condition $q(\delta) \Vdash H(\delta) = i(\delta)$. Put $q(\delta) = p(\nu(\delta))$. We include the functions $i(\delta), \nu(\delta), t(\delta) = \min\{\gamma \geq \delta \colon \xi < \delta \text{ implies } \tau(p(\xi)) \leq \gamma\}$ into M.

If $\theta, i < \omega_2$ are given, enumerate as $\{q_{\theta i j} : j < \omega_2\}$ a maximal incompatible system of conditions $q \in P$ such that $\theta^q = \theta$, there is a $\xi(q) < \omega_3$ with the property that whenever $\theta^{q(\delta)} = \theta$, $i(\delta) = i$, and $q(\delta), q$ are compatible, then $h(\delta) \leq \xi(q)$. Such systems exist by Lemma 1. Select $c < \omega_3$ so large that $\tau(q_{\theta i j}), \xi(q_{\theta i j}) < c$ for $\theta, i, j < \omega_2$. Include c into M. Add also the function $K(i, \nu, \alpha) = \min\{\gamma \colon i(\gamma) = i, q(\gamma)|\alpha = p(\nu), h(\gamma) \geq \alpha\}$. We now have the model M, and so can apply the claim that for some $\delta \in B$, $M_\delta \prec M$. Assume that $i(\delta) = i$, $q(\delta)|\delta = p(\zeta)$ then, as t is in M, $\zeta < \delta$. Assume that ζ is in $[\alpha, \beta)$, the μ-th complementary interval of C'_δ. Assume that $e_{\alpha,\beta}(\nu) = \zeta$, and $\xi < \omega_2$ satisfies $G(\xi) = (i, \mu, \nu)$. Put $\theta = \theta^{p(\zeta)}$. Assume that the ξ-th element of C'_δ is ε. What happened at the construction of $g_\delta(\varepsilon)$? If there exists a γ such that $\varepsilon < h(\gamma)$, $i(\gamma) = i, \theta^{p(\nu(\gamma))} = \theta$ and $p(\nu(\gamma)), p(\zeta)$ are compatible, then $\{\gamma, \delta\} \in X$ for the least such γ, and a common extension of $p(\nu(\gamma))$ and $q(\delta)$ forces that $H(\delta) = H(\gamma) = i$, a contradiction. We can, therefore, assume that for every γ if $i(\gamma) = i$, $\theta^{p(\nu(\gamma))} = \theta$, $p(\nu(\gamma))$, $p(\zeta)$ are compatible, then $h(\gamma) < \varepsilon$. Then, there exists a $j < \omega_2$, such that $q_{\theta i j}$, $p(\zeta)$ are directly compatible, so $h(\delta) < c = c_\delta$, a contradiction. ∎

To conclude the proof of the Theorem, we are going to show that X on every bounded subset of ω_3 is countably chromatic. This suffices, by Lemma 2. If $\beta < \omega_3$, a good coloring $H\colon \beta \to \omega$ is *suitable*, if for every limit $\delta \leq \beta$, the set $\omega - \{H(g_\delta(\xi)) \colon \xi \in C'_\delta\}$ is infinite.

Lemma 4. *If $\beta < \alpha < \omega_3, H\colon \beta \to \omega$ is suitable, H' is a coloring of some finite subset of $[\beta, \omega_3)$ such that $H \cup H'$ is a good coloring, then there is a suitable extension of H to α which is compatible with H'.*

Proof. By induction on α. To pass from α to $\alpha + 1$, assign a color to α, compatible with H', and use the Lemma for α. If α is limit, let $\{\gamma_\xi \colon \xi < \operatorname{otp}(C'_\alpha)\}$ be the increasing enumeration of C'_α, with $\gamma_0 = 0$. Suppose that $\gamma_\zeta \leq \beta < \gamma_{\zeta+1}$. As H is suitable, $A = \omega - \{H(g_\alpha(\gamma_\xi)) : \xi \leq \zeta\} - \operatorname{Ran}(H')$ is infinite. Choose $k \in A$. Using the Lemma repeatedly, we extend H from β to $\gamma_{\zeta+1}$, then from $\gamma_{\zeta+1}$ to $\gamma_{\zeta+2}$, and so on, but coloring the vertices $g_\alpha(\gamma_\varepsilon)$ for $\zeta < \varepsilon < \operatorname{otp}(C'_\alpha)$, not in the domain of H', only with k. For a limit ordinal $\xi \leq \operatorname{otp}(C'_\alpha)$, $\{H(g_\alpha(\gamma_\tau)) \colon \tau < \xi\}$ contains only one element of A. The induction step is possible as $g_\alpha(\gamma_\xi)$ is connected by no edge into points in γ_ξ as $\gamma_\xi \leq h(g_\alpha(\gamma_\xi))$. ∎

We notice that under $2^{\aleph_2} = \aleph_3$, Conjecture 3 is equivalent to the statement that $G(\omega_2)$ can be embedded into $G(\omega_3)$. Various questions can be asked about the embeddability of $G(\kappa)$ into $G(\lambda)$ for different κ, λ. As $G(\omega)$ contains a complete graph with 2^ω vertices, it can not be embedded into $G(\kappa)$ for $\kappa \geq \omega_2$, and $G(\kappa)$ embeds into $G(\omega)$ iff $2^\kappa = 2^\omega$. $G(\omega)$ embeds into $G(\omega_1)$ iff there is a family of 2^ω almost disjoint $\omega_1 \to \omega$ functions. For $\kappa < \lambda$, $G(\kappa)$ trivially embeds into $G(\lambda)$, if $\mathrm{Chr}(G(\kappa)) = \omega$, and this holds if κ is measurable. If $2^\kappa = \kappa^+$, and κ is weakly compact, then $G(\kappa)$ embeds into $G(\kappa^+)$, so it is reasonable to ask the following question.

Conjecture 4. *If κ is weakly compact, then* $\mathrm{Chr}(G(\kappa)) = \omega$.

Acknowledgment. The author is grateful to the referee for several comments.

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