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Note on a Problem Concerning Nuclei

G. KORCHMÁROS and F. MAZZOCCA

ABSTRACT

In a previous paper [3], we gave a complete classification for pairs $(\mathcal{N}(\mathcal{B}), \mathcal{B})$ where $\mathcal{N}(\mathcal{B})$ is the nuclei set of a set $\mathcal{B}$ of $q+1$ non collinear points contained in the union of two lines in a Desarguesian plane of order q. Our method involved some properties of projectivity groups over a Galois field and hence does not work in an arbitrary plane. In this paper we shall adopt a different approach based on collineations instead of projectivities, and show how to apply the technique developed by Hering in [2]. We are able to extend some substantial results given in [3] to a non-desarguesian plane under the condition that every nucleus of $\mathcal{B}$ is the center of a central collineation interchanging the lines containing $\mathcal{B}$, see Proposition 2. However the classification question seems well beyond present knowledge on non-desarguesian planes.

1. Introduction

Let $\mathcal{B}$ be any set of $n+1$ points in a projective plane Π of order n. A *nucleus* of $\mathcal{B}$ is a point x such that each line of Π through x meets $\mathcal{B}$ exactly once. It follows that x is not on $\mathcal{B}$.

Fix two distinct lines L_1 and L_2 in Π, and take a set $\mathcal{B}$ of $n+1$ non-collinear points contained in $L_1 \cup L_2$ such that $z = L_1 \cap L_2 \in \mathcal{B}$ and $\mathcal{B}$ admits some nucleus. Let $\mathcal{N}(\mathcal{B})$ denote the set of all nuclei of $\mathcal{B}$.

For $i = 1, 2$, let $\mathcal{B}_i$ denote the set of common points of $\mathcal{B}$ and L_i, and put $\overline{\mathcal{B}}_i = (L_i - \mathcal{B}_i) \cup \{z\}$. A simple but very useful observation is as follows:

Propositon 1. *Assume that the point B is the center of an involutorial elation* $\mathbf{u}$ *which interchanges L_1 with L_2. Then $B \in \mathcal{N}(\mathcal{B})$ if and only if* $\mathbf{u}$ *interchanges $\mathcal{B}_1$ with $\overline{\mathcal{B}}_2$ (and $\mathcal{B}_2$ with $\overline{\mathcal{B}}_1$).*

This observation has suggested us to use group theory to investigate $\mathcal{N}(\mathcal{B})$. There is a well developed theory of collineation groups containing involutorial elations, and our purpose is to use it in order to extend some previous results on pairs $(\mathcal{B}, \mathcal{N}(\mathcal{B}))$ to non-desarguesian planes of even order. We shall see that this can be done under the hypothesis that

(*) *each nucleus is the center of an involutorial elation interchanging L_1 with L_2.*

Nevertheless, there could exist some pairs $(\mathcal{B}, \mathcal{N}(\mathcal{B}))$ which do not occur in the Desarguesian plane, as Proposition 2 suggests: In a Desarguesian plane $\mathcal{N}(\mathcal{B})$ does not coincide with a subplane, see Proposition 5 of [3], but this might not hold in some non-desarguesian planes.

Of course hypothesis (*) is valid if every point not on either line is the center of an involutorial elation interchanging L_1 with L_2. If this is the case, Π is a dual Bol plane with improper line z and improper points L_1 and L_2 on the co-ordinate axes, see ([4], VI.41). Recall that, by a result of Kallaher (see, ibid. Theorem 41.10), the Bol planes of even order are near-field planes.

Our terminology is standard, see [1] and [2]. Moreover, for a point-set X of Π we shall use the term of an *i-secant* to denote a line of Π which meets X in exactly i points. Also, X is said to be of type $[x_1, x_2, \ldots, x_s]$ if $0 \le x_1 < x_2 < \ldots < x_s$ are integers such that the only values of i for which X admits an i-secant are just $x_1, x_2, \ldots, x_s$.

2. A theorem on the configuration of $\mathcal{N}(\mathcal{B})$

According to (*), let $\mathbf{u}$ denote the (unique) involutorial elation with center $u \in \mathcal{N}(\mathcal{B})$ interchanging L_1 with L_2. Clearly uz is the axes of $\mathbf{u}$. By Proposition 1, the product of any two involutorial elations whose center are in $\mathcal{N}(\mathcal{B})$ preserves $\mathcal{B}$. Therefore, the collineation group S generated by the products $\mathbf{u}_1\mathbf{u}_2$ where u_1 and u_2 range over $\mathcal{N}(\mathcal{B})$ is a subgroup of the group of all collineations preserving $\mathcal{B}$.

From now on we assume that

(i) *G contains an involutorial elation with center z.*

This condition is certainly satisfied if $\mathcal{B}$ has two nuclei u_1 and u_2 on a line L through z. Indeed, L is the common axis of $\mathbf{u}_1$ and $\mathbf{u}_2$, hence their product is an involutorial elation with center z. Furthermore, assume that

(ii) *G contains involutorial elations whose axes are different and do not pass through z.*

Observe that if $\mathcal{N}(\mathcal{B})$ is not on a line through z, then (ii) holds.

We have shown that under our assumptions there are three non collinear points u, v, z such that each of them is the center of an involutorial elation of G and one of them, z, is left invariant by G.

Now we may apply, up to duality, some results contained in [2]. To do this we need some notation from that paper. For any point x and any line X, $G(x, X)$ denotes the subgroup of G consisting of all involutorial elations with center x and axis X; $G(x, x)$ is the subgroup of G consisting of all involutorial elations with center x; $G(X, X)$ is the subgroup of G consisting of all involutorial elations with axis X. Furthermore,

$$\mathfrak{U} = \{x \mid G(x,x) \neq 1\}, \quad \mathfrak{Z} = \{X \mid G(X,X) \neq 1\}, \quad T = G(z,z),$$

$$C = \mathfrak{C}_S T = \{g \mid gt = tg, \text{ for all } t \in T\}, \; \Omega = \{X \in \mathfrak{Z} \mid G(X,X) \not\subseteq T\},$$

$$\mathfrak{K} = \{G(X,z) \mid X \in \Omega\}, \quad q = [G(Z,Z) : G(Z,z)] \text{ with } Z \in \Omega,$$

$$K = \{g \in G \mid g(X) = X, \text{ for all } X \in \Omega\}, \; \mathfrak{F} = \{X \mid z \notin X, |X \cap \mathfrak{U}| \geq 2\}.$$

Observe that

(I) $\mathfrak{U} = \mathcal{N}(\mathcal{B}) \cup \{z\}$.

(II) Ω is the set of all lines through z which meet $\mathcal{N}(\mathcal{B})$.

(III) $\mathfrak{F}$ is the set of all secants of $\mathcal{N}(\mathcal{B})$ missing z.

Under our assumptions q is equal to 2, since $\mathbf{uv} \in G(Z, z)$ for any two $\mathbf{u}, \mathbf{v} \in G(Z, Z)$ with $Z \in \Omega$. By the observation made after (3.12) of [2, p. 39] T has order 2^{2h}, and from (3.9) of [2], $4 \not| |S/T|$ follows. By (4.2) of [2] T is sharply transitive on $\mathfrak{F}$, hence $\mathfrak{F}$ has size 2^{2h}. Also, from (4.4) of [2] we get $|\mathcal{N}(\mathcal{B})| = 2^h t$ where $t = |\Omega|$. By (4.5) and (4.6) of [2], if a line L through z meets $\mathcal{N}(\mathcal{B})$ and u is a point on $\mathcal{N}(\mathcal{B})$, then the number of all secants of $\mathcal{N}(\mathcal{B})$ at u and the size of $L \cap \mathcal{N}(\mathcal{B})$ are both equal to 2^h. By counting in two different ways all flags (u, F), with $u \in \mathcal{N}(\mathcal{B})$ and $F \in \mathfrak{F}$, we see that every secant belonging to $\mathfrak{F}$ is a t-secant of $\mathcal{N}(\mathcal{B})$.

Observe that $|L \cap \mathcal{N}(\mathcal{B})| = 2^h$ implies that $|G(Z,Z) - G(Z,z)| = 2^h$ for each $Z \in \Omega$. As $q = 2$, it follows from this that $|G(Z,z)| \geq 2^h$ holds, and hence T contains at least $t(2^h - 1) + 1$ elements. On the other hand $|T| = 2^{2h}$. It turns out that either $t \leq 2^h - 1$, or $t = 2^h + 1$. In the latter case $\mathcal{N}(\mathcal{B}) \cup \{z\}$ is a subplane of order 2^h.

We summarize our results in

Propositon 2. *If $\mathcal{B}$ admits three non-collinear nuclei, two of them on a line through z, then either $\mathcal{N}(\mathcal{B}) \cup \{z\}$ is a subplane of order 2^h, or $\mathcal{N}(\mathcal{B})$ is a set of size $2^h t$ and of type $[0, 1, t, 2^h]$. In the latter case the t-secants are concurrent at z.*

3. A condition for $\mathcal{N}(\mathcal{B})$ not to be a subplane

For a secant $F \in \mathfrak{F}$, let H denote the subgroup of S which leaves F invariant. Clearly H contains every involutorial elation with center on F. By (3.11) of [2], S splits over T. As T is sharply transitive on $\mathfrak{F}$, H turns out to be a complement of T in S. To obtain some further significant results, Hering limited himself to investigate the case of

(iii) $O(S/C)$ *abelian.*

It should be noticed that it is not known whether this condition is satisfied in a nearfield plane or any other non-desarguesian plane. However, we are able to prove that if (iii) holds then $\mathcal{N}(\mathcal{B})$ is can not be a subplane of Π.

By (3.8) of [2], $K = C = T$, and H is a dihedral group of order twice an odd number. Thus the elements of order 2 of H are pairwise conjugate under H. We see that the elements of order 2 in H are exactly the non-trivial involutorial elations with center on F. Let D denote the cyclic subgroup of index 2 of H. Then each element of D is the product of two involutorial elations with center on F.

Let Γ be the subgroup of index 2 of S consisting of all collineations of S which leave L_i invariant, i.e. which preserve both $\mathcal{B}_i$'s. Consider the action of Γ on L_i. Clearly, $b_i = F \cap L_i$ is the unique fixed point of D on L_i. Let T_i be the the group of all involutorial elations with axis L_i. Note that $T_1 \cong T_2$ because $T_2 = \mathbf{u}T_1\mathbf{u}$ for any $u \in \mathcal{N}(\mathcal{B})$. Furthermore, T_i is the subgroup of Γ which fixes L_i pointwise. Put $\mid T_i \mid = 2^s$. We have $h \geq s$, since the involutorial elations with a common axis $L \in \Omega$ form a subgroup of order 2^h having trivial intersection with T. Therefore, T/T_i has order 2^k with $k \geq$h. We see that Γ acts on L_i as a permutation group $\overline{\Gamma}$ of order 2^kt. Let $\mathbf{B}^i$ denote the orbit of b_i under $\overline{\Gamma}$. Since Γ is transitive on $\mathfrak{F}$, $\mathbf{B}^i$ consists of all points $F \cap L_i$ where F ranges over $\mathfrak{F}$. Any other orbit $\mathbf{O}^i$ has length $2^k t$ and $\overline{\Gamma}$ acts on $\mathbf{O}^i$ as a regular permutation group. There exist $m = (q-2^k)/2^k t$ orbits of the latter type, thus $\mathbf{B}^i \cup \mathbf{O}_1^i \cup \ldots \cup \mathbf{O}_m^i$ is the partition of $L_i - \{z\}$ by orbits under $\overline{\Gamma}$. Observe that any $\mathbf{u} \in \mathbf{U}$ interchanges $\mathbf{B}^1$ with $\mathbf{B}^2$ and $\mathbf{O}_j^1$ with $\mathbf{O}_j^2$ $(1 \leq j \leq m)$. From this we get that $\mathcal{N}(\mathcal{B})$ is a set of size $2^h t$ and of type $[0, 1, t, 2^h]$.

As in Section 3 of [3], we point out some special cases corresponding to special structures of Γ:

1. *If Γ is trivial, then $\mathcal{N}(\mathcal{B})$ consists of a single point.*
2. If $\Gamma = T$, then $\mathcal{N}(\mathcal{B})$ is a set of 2^h points on a line through z.

Assume that each one of the conditions (i), (ii), (iii) holds. Then the first possibility in Proposition 2 does not occur. Furthermore,

3. *If T is trivial, then $\mathcal{N}(\mathcal{B})$ is a set of t poins on a line missing z.*
4. *If $t = 2^h - 1$, then $\mathcal{N}(\mathcal{B})$ together with $\mathbf{B}^1 \cup \mathbf{B}^2 \cup \{z\}$ form a projective subplane.*

References

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G. Korchmáros
Department of Mathematics,
University of Basilicata,
via N. Sauro 85,
I-85100 Potenza, Italy

F. Mazzocca
Department of Mathematics,
University of Napoli,
via Mezzocannone 8,
I-80134 Napoli, Italy