COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
60. SETS, GRAPHS AND NUMBERS, BUDAPEST (HUNGARY), 1991

Counting of Path-Like Objects in a Rectangular Array

D. M. KULKARNI*

ABSTRACT

Let $X = [X_{ij}]_{\substack{1\le i\le m\\ 1\le j\le n}}$ be an m by n matrix of indeterminates over a field K. Abhyankar defines the index of a monomial in X_{ij} to be the largest k such that the principal diagonal of some k by k minor of X divides the given monomial. Abhyankar has given a formula for counting the set of monomials in X_{ij} of degree V satisfying a certain set of index conditions. This formula gives the Hilbert polynomial of a certain generalized determinantal ideal which can be viewed as a polynomial in V with rational coefficients. We develop a combinatorial map from this set of monomials to the set of special subsets of tuples of nonintersecting paths, called p-tuples of vertical pathoids here, in the m by n rectangular lattice of points. A path from (a, n) to (m, b) in a rectangular m by n array is obtained by moving either left or down at each point. The point where the path turns from down to left is called its node. A special subset of a path containing its nodal set is called a vertical pathoid. Exploiting the linear independence of a family of binomial coefficients, we get formulae counting sets of p-tuples of vertical pathoids of a given size in a rectangular lattice. This helps us to interpret "coefficients" of Hilbert polynomials of generalized determinantal ideals combinatorially.

* This work was supported by the Oakland University Research Fellowship.

1. Introduction

Given an m by n rectangular array of points (where the vertices are labelled like a matrix), a path from $A(a,n)$ to $B(m,b)$ moves either left or down at each vertex. (See Figure 1). Given two sequences of integers $1 \le a_1 < a_2 < \ldots < a_p \le m$ and $1 \le b_1 < b_2 < \ldots < b_p \le n$; we denote the ordered pair of the sequences $(a_1a_2\ldots a_p,\ b_1b_2\ldots b_p)$ by $\tilde{a}$ and refer to it as a bivector $\tilde{a}$ of length p bounded by (m,n). We refer to any p-tuple of paths in the rectangular m by n array where the ith path goes from (a_i,n) to (m,b_i) for $i = 1,2,\ldots p$ as a p-tuple of paths based on a bivector $\tilde{a}$. The path-like objects we count here are called p-tuples of vertical pathoids of a fixed size which are certain subsets of the p-tuples of nonintersecting paths based on a bivector $\tilde{a}$ of length p. Our proof uses a combinatorial map from the special set of monomials in mn indeterminates X_{ij} to the set of p-tuples of vertical pathoids, and an algebraic manipulation of Abhyankar's formula from [1] (stated in (2.1) here) for counting the above special set of monomials.

The principle of the counting goes as follows: Let a family of polynomials (when V is viewed as an indeterminate over the field of rationals $\mathbb{Q}\{f(V)\}_{V=0}^{\infty}$ give the cardinality of a family of finite sets $\{X_V\}_{V=0}^{\infty}$. Let $\{Y_E\}_{E=0}^{\infty}$ be a family of pairwise disjoint finite sets. We set up a map Ψ from X_V to the set $\bigcup_{E=0}^{\infty} Y_E$ so that for each y in $\bigcup_{E=0}^{\infty} Y_E$, $\Psi^{-1}(y)$ contains $\lambda(V,E)$ elements where $\lambda(V,E)$ depends only upon V and E and not on y. Thus we get

$$|X_V| = \sum_{E=0}^{\infty} |Y_E|\lambda(V,E).$$

If we can write $f(V) = \sum_{E=0}^{\infty} H_E\lambda(V,E)$ where $\{\lambda(V,E)\}_{E=0}^{\infty}$ is a family of $\mathbb{Q}$-linearly independent polynomials in $\mathbb{Q}[V]$ we get

$$|Y_E| = H_E \quad \text{for} \quad E = 0,1,2,\ldots\ .$$

We have used the above principle of a combinatorial map and comparing coefficients, in [4] to count the number of p-tuples of nonintersecting paths based on a bivector $\tilde{a}$ having fixed number of nodes (definition in section 2 here) in the m by n rectangular array. The interesting part of this technique is how easily the original counting of monomials affords a decent expression for counting a complicated set of path-like objects, which otherwise would have been a tedious task.

These results have arisen from Abhyankar's questions about the coefficients of the Hilbert polynomials of the generalized determinantal ideals. The generalized determinantal ideals $I(p, \tilde{a})$ he considered in [1], are generated by the i by i minors from the first a_i rows or b_i columns ($i = 1, 2, \ldots, p$) of an m by n matrix X of indeterminates as well as all the $p+1$ by $p+1$ minors from X, in the polynomial ring $K[X]$ where K is a field. In [1], he shows that the special set of monomials (defined in section 2 here) of degree V forms a K-free basis for the degree V component of the quotient ring $\frac{K[X]}{I(p,\tilde{a})}$ and establishes that these ideals are Hilbertian i.e. their Hilbert function and Hilbert polynomial coincide for all nonnegative integers. It is shown by explicitly writing this Hilbert function as a polynomial in V with rational coefficients. Instead of using usual basis $\{V^E\}_{E=0}^{\infty}$ for that, he uses a basis of twisted binomial coefficients $\{\binom{V+E}{E}\}_{E=0}^{\infty}$ to write these polynomials. This makes the "coefficients" of Hilbert polynomials a bit hard to be interpreted. Our results here and in [4] interpret some of these coefficients. If we look at the expression in (2.3) as it stands, it is not easy to see that the expression gives a nonnegative integer in general. It follows immediately from our results. It would have been nice, had this counting expression been a single determinant like those in [2], instead of the sum of several determinants. We observe in Example III (section 3) that in fact, each individual determinant in the sum in (2.3) may not be nonnegative, although the sum is. For other recent interesting lattice-path counting results, an interested reader may look at [3] or [5].

2. Preliminaries and Abhyankar's Formula

Given an m by n rectangular array of points (where the vertices are labelled like a matrix), a path from $A(a, n)$ to $B(m, b)$ moves either left or down at each vertex. The points where it goes from down to left are called its nodes. In Figure 1, for $m = 7$ and $n = 6$, we illustrate a path from $(3, 6)$ to $(7, 2)$. The points (4,5) and $(6, 4)$ are the nodes of this path in Figure 1. If a path from (a, n) to (m, b) has nodes at (x_1, y_1), $(x_2, y_2), \ldots, (x_e, y_e)$ and $x_1 < x_2 < \ldots < x_e$, its vertical part is defined to be

$$\left(\bigcup_{i=1}^{e} \{(x, y_i) : x_{i-1} < x \leq x_i\}\right) \cup \{(x, b) : x_e < x \leq m\}$$

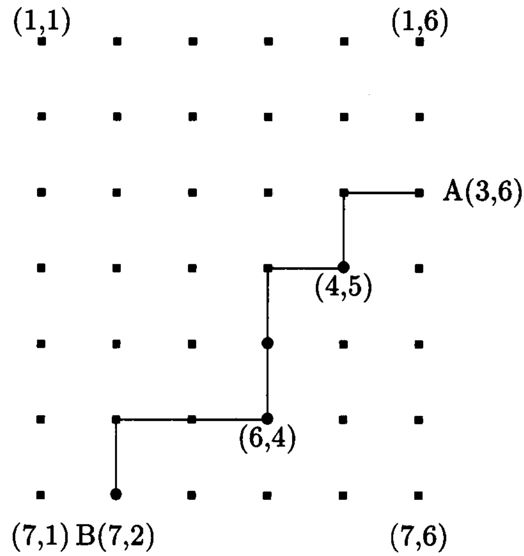

Figure 1.

where $x_0 = a$. The encircled points $\{(4,5)\ (5,4),\ (6,4)\ (7,2)\}$ are said to form the vertical part of the path. Note that the vertical part of the path includes its nodes. A subset of the vertical part of the path containing its nodes is called a vertical pathoid.

If we take $A_1(a_1, n), A_2(a_2, n), \ldots, A_p(a_p, n)$ and $B_1(m, b_1), B_2(m, b_2), \ldots, B_p(m, b_p)$ where $a_1 < a_2 < \ldots < a_p$ and $b_1 < b_2 < \ldots < b_p$; we look at the ordered p-tuple of nonintersecting paths $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ where σ_i is a path going from A_i to B_i for $i = 1, 2, \ldots, p$; and σ_i and σ_j do not share a vertex for $i \neq j$, $1 \leq i,\ j \leq p$. This is a p-tuple of nonintersecting paths based on a bivector $\tilde{a} = (a_1 a_2 \ldots a_p, b_1 b_2 \ldots b_p)$. We fix this bivector $\tilde{a}$ for the rest of the paper. The vertical part of p-tuple of paths is the union of the vertical parts of the component paths, and the nodes of the p-tuple consists of the union of the nodes of the component paths. A p-tuple of vertical pathoids is a subset of the vertical part of an ordered p-tuple of nonintersecting paths containing the nodes of the p-tuple. It is important to see that a p-tuple of vertical pathoids is in fact a set having a unique representation as an ordered p-tuple of vertical pathoids based on a bivector $\tilde{a}$. The encircled points in Figure 2 illustrate a 3-tuple of vertical pathoids in the 6 by 8 rectangle and its size is 7.

Let m and n be positive integers. Let $\text{rec}(m, n)$ denote an m by n rectangular array i.e. the set of all ordered pairs of integer entries (i, j) where

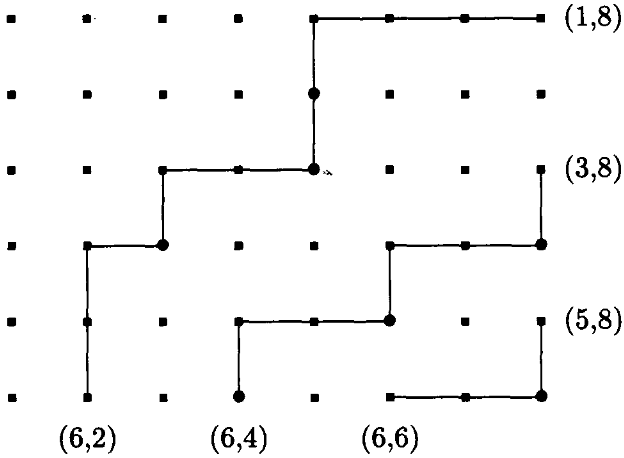

Figure 2.

$1 \le i \le m$ and $1 \le j \le n$. A subset S of $\text{rec}(m,n)$ is said to be of index k if S contains a sequence of length k given by (x_1, y_1), $(x_2, y_2), \ldots, (x_k, y_k)$ where $x_1 < x_2 < \ldots < x_{k-1} < x_k$ and $y_1 < y_2 < \ldots < y_{k-1} < y_k$; but no such sequence of length $k+1$. Note that only an empty set has index zero. A monomial on the $\text{rec}(m,n)$ can either be viewed as a map from $\text{rec}(m,n)$ to the set of nonnegative integers or a multiset on the set of ordered pairs in the support i.e. the variables which actually appear. The index of a monomial on $\text{rec}(m,n)$ is the index of its support. The special set of monomials we are looking for is denoted by $\text{mon}((m,n),p,\tilde{a},V)$. For positive integers m,n,p; a nonnegative integer V and a bivector $\tilde{a}$ of length p bounded by (m,n), we describe $\text{mon}((m,n),p,\tilde{a},V)$ to be the set of all monomials on $\text{rec}(m,n)$ of degree V with index at most p, and when restricted to either the first $a_i - 1$ rows or the first $b_i - 1$ columns of the rectangular array, index is at most $i-1$ for $i = 1, 2, \ldots, p$. The set of all such monomials in $\text{rec}(m,n)$ is denoted by $\text{mon}((m,n),p,\tilde{a})$.

For convenience, for an integer A and an indeterminate V over $\mathbb{Q}$, we define the usual binomial coefficient as

$$\binom{V}{A} = \begin{cases} \frac{V(V-1)\ldots(V-A+1)}{A!} & A \ge 0 \\ 0 & A < 0 \end{cases}$$

and the twisted binomial coefficient as

$$\begin{bmatrix} V \\ A \end{bmatrix} = \begin{cases} \frac{(V+1)(V+2)\ldots(V+A)}{A!} & A \ge 0 \\ 0 & A < 0. \end{cases}$$

In [1, Theorem 9.8, p. 306], Abhyankar has shown that

$$|\mathrm{mon}\,((m,n),p,\tilde{a},V)| = \sum_{D\in\mathbb{Z}} (-1)^D F_D\,((m,n),p,\tilde{a}) \begin{bmatrix} V \\ C'-D \end{bmatrix} \tag{2.1}$$

where $C' = \sum_{i=1}^{p}(m-a_i) + \sum_{j=1}^{p}(n-b_j) + p - 1$, and for each D in $\mathbb{Z}$,

$$F_D\,((m,n),p,\tilde{a}) = \sum_{E\in\mathbb{Z}} (-1)^{R-E} \binom{E}{D+E-R} H_E^*\,((m,n),p,\tilde{a}) \tag{2.2}$$

where $R = \sum_{i=1}^{p}(m-a_i)$, and for each E in $\mathbb{Z}$ and $e = (e_1, e_2, \ldots, e_p)$ a p-tuple of integers,

$$H_E^*\,((m,n),p,\tilde{a}) = \sum_{e_1+e_2+\ldots+e_p=E} \left\{ \prod_{i=1}^{p} \begin{bmatrix} n-b_i \\ e_i \end{bmatrix} \right\} \det_{1\le i,j\le p} \binom{m-a_j+j-i}{m-a_j-e_i} \tag{2.3}$$

where $\det U_{i,j}$ denotes the determinant of a p by p square matrix whose ijth entry is $U_{i,j}$. If we write an expression in (2.1) as

$$|\mathrm{mon}\,((m,n),p,\tilde{a},V)| = \sum_{E\in\mathbb{Z}} \left\{ \sum_{D\in\mathbb{Z}} (-1)^{D-R+E} \begin{bmatrix} V \\ C'-D \end{bmatrix} \binom{E}{D+E-R} \right\} H_E^*\,((m,n),p,\tilde{a})$$

we can use

$$\sum_{D\in\mathbb{Z}} (-1)^{D-R+E} \begin{bmatrix} V \\ C'-D \end{bmatrix} \binom{E}{D+E-R} = \begin{bmatrix} V-E \\ S+p-1+E \end{bmatrix}$$

where $S = \sum_{i=1}^{p}(n-b_i)$, and get from (2.1),

$$|\mathrm{mon}\,((m,n),p,\tilde{a},V)| = \sum_{E\in\mathbb{Z}} H_E^*\,((m,n),p,\tilde{a}) \begin{bmatrix} V-E \\ S+p-1+E \end{bmatrix}. \tag{2.4}$$

3. Combinatorial Map and Applications

In [4], we have shown that there is a map φ from $\text{mon}((m,n),p,\tilde{a},V)$ to the set of p-tuples of nonintersecting paths in $\text{rec}(m,n)$ based on a bivector $\tilde{a}$ which produces a counting formula for the set of p-tuples of nonintersecting paths in $\text{rec}(m,n)$ based on a bivector $\tilde{a}$ having fixed number of nodes. We illustrate this map here by an Example.

Example I.

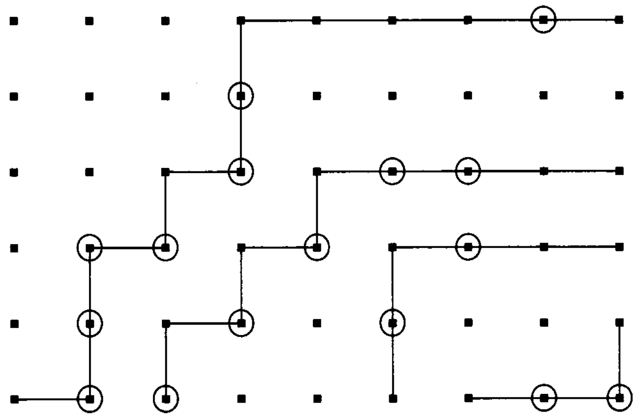

Figure 3.

Let $m = 6$, $n = 9$ and $\tilde{a} = (1\,3\,4\,5,\ 1\,3\,6\,7)$. Let us look at a monomial

$$M = X_{18}X_{24}^2X_{34}X_{36}^2X_{37}X_{42}^3X_{43}X_{45}^2X_{47}X_{52}X_{54}X_{56}X_{62}X_{63}X_{68}^2X_{69}$$

where its support i.e. the set of variables which appear, corresponds to

$$T = \{(1,8),(2,4),(3,4),(3,6),(3,7),(4,2),(4,3),(4,5),(4,7),(5,2),(5,4),\\ (5,6),(6,2),(6,3),(6,8),(6,9)\}.$$

Note that M lies is $\text{mon}((6,9),4,\tilde{a})$. The method of associating 4-tuple of paths to T based on $\tilde{a}$ is described as follows:

Start at $A_4(5,9)$ and move left along the fifth row until you come to a vertex of a column that has a vertex from T below it and then go down along that column until you exhaust all of the vertices from T, and at the downmost vertex of T there, start moving left until you hit either a column of B_4 or a column which has a vertex of T below it. If you get a column of B_4, go down to B_4. If you hit a column which has a vertex of T below it, repeat the above process until you reach B_4. In the example, $(6,9)$ lies below $(5,9)$ and a path from A_4 has to go to $(6,9)$ and turn left and reach B_4. Note that $(6,9)$ is a node of the path.

In the next step, remove those vertices from T which are on a path from A_4 to B_4 to get a set T_1 and find a path from A_3 to B_3 associated with T_1 following the above procedure. Now our T_1 is

$$T_1 = \{(1,8),(2,4),(3,4),(3,6),(3,7),(4,2),(4,3),(4,5),(4,7),(5,2),(5,4),(5,6),(6,2),(6,3)\}$$

and we start moving left from A_3 $(4,9)$ along the fourth row and after crossing $(4,7)$ we hit the column of B_3 and just go down to B_3. Note that there is no node on this path.

By removing vertices of T_1 lying on a path from A_3 to B_3, we get

$$T_2 = \{(1,8),(2,4),(3,4),(3,6),(3,7),(4,2),(4,3),(4,5),(5,2),(5,4),(6,2),(6,3)\}.$$

Applying the above procedure, we get a path from A_2 to B_2 and then from A_1 to B_1. The set of nodes for this 4-tuple of nonintersecting paths $\varphi(M)$ based on $\tilde{a}$ is $\{(3,4),(4,3),(6,2),(4,5),(5,4),(6,9)\}$. Note that $(6,3)$ is not a node because a path does not "turn" there.

We proved in [4] the following

Theorem 1. *Given positive integers m,n and p and a bivector $\tilde{a}$ of length p bounded by (m,n), to each monomial M is* $\mathrm{mon}((m,n),p,\tilde{a})$, *we can associate the unique p-tuple of nonintersecting paths $\varphi(M)$ based on $\tilde{a}$. The set of nodes of the p-tuple decides the p-tuple of paths completely.*

We want to modify the above map a little bit to get a map from $\mathrm{mon}((m,n),p,\tilde{a})$ to the set of all p-tuples of vertical pathoids based on $\tilde{a}$. Let Ψ be a map which associates to each monomial M in $\mathrm{mon}((m,n),p,\tilde{a})$, the part of M lying on the vertical part of $\varphi(M)$. It is clear that $\Psi(M)$ is a p-tuple of vertical pathoids based on $\tilde{a}$, as $\varphi(M)$ is a p-tuple of nonintersecting paths based on $\tilde{a}$. For M in the above example,

$$\Psi(M) = \{(2,4),(3,4),(4,3),(5,2),(6,2),(4,5),(5,4),(6,3),(5,6),(6,9)\}$$

is a 4-tuple of vertical pathoids based on $\tilde{a}$.

From Theorem 1, we can easily see

Theorem 2. *Given positive integers m, n and p; and a bivector $\tilde{a}$ of length p bounded by (m, n), to each monomial M in* $\mathrm{mon}((m, n), p, \tilde{a})$, *we can associate a unique p-tuple of vertical pathoids $\Psi(M)$ based on $\tilde{a}$.*

It is important to note that in a p-tuple of vertical pathoids, some "components" may be empty sets. We illustrate it by the following example.

Example II.

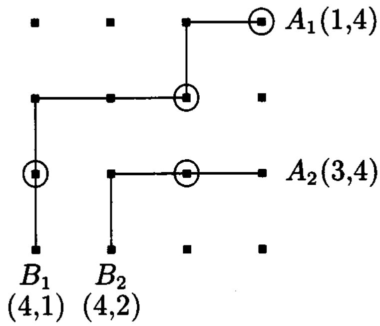

Figure 4.

To a monomial $M' = X_{14}^2 X_{23}^3 X_{31} X_{33}^2$, the above map Ψ will associate a tuple of vertical pathoids given by $\{(2,3),(3,1)\}$. Note that $\{(2,3),(3,1)\}$ is a vertical pathoid lying on a path from A_1 to B_1, and the contribution from a path from A_2 to B_2 to it is an empty set.

The p-tuple of nonintersecting paths in $\varphi(M)$ for each monomial M in $\mathrm{mon}((m, n), p, \tilde{a})$ is completely determined by its set of nodes. Since $\Psi(M)$ contains the nodes of $\varphi(M)$, it is clear that $\varphi(M)$ is the unique p-tuple of nonintersecting paths associated with $\Psi(M)$ under φ. Thus we have

Lemma 3. *For positive integers m, n and p; and a bivector $\tilde{a}$ of length p bounded by (m, n), for each monomial M in* $\mathrm{mon}((m, n), p, \tilde{a})$, *we have* $\varphi(\Psi(M)) = \varphi(M)$.

If we look at any p-tuple of vertical pathoids τ based on $\tilde{a}$, we know that τ can have at most $\sum_{i=1}^{p}(m - a_i) = R$ vertices in it; since the ith vertical pathoid lying on a path from (a_i, n) to (m, b_i) has at most $m - a_i$ elements for $i = 1, 2, \ldots, p$. For a given p-tuple of vertical pathoids τ based on $\tilde{a}$, a monomial M in $\mathrm{mon}((m, n), p, \tilde{a})$ will go to τ under Ψ if the part of the support of M lying on the vertical part of $\varphi(M)$ is precisely τ. If τ contains E vertices, the number of monomials M in $\mathrm{mon}((m, n), p, \tilde{a}, V)$

which go to τ under Ψ is $\left[{V-E \atop S+p-1+E} \right]$. It is clear because E variables from τ have to appear in a monomial of degree V, and the support of these monomials will contain possibly some variables from the $\sum_{i=1}^{p}(n-b_i+1)$ variables of the "horizontal" (i.e. not vertical) part of the associated p-tuple of nonintersecting paths under φ. Thus we have

Lemma 4. *For positive integers m, n, p and a bivector $\tilde{a}$ of length p bounded by (m, n), for a p-tuple of vertical pathoids τ based on $\tilde{a}$ containing E vertices, there are $\left[{V-E \atop S+p-1+E} \right]$ monomials of* $\mathrm{mon}((m,n),p,\tilde{a},V)$ *which go to τ under Ψ.*

Let pathoid $((m,n),p,\tilde{a},E)$ denote the set of p-tuples of vertical pathoids based on $\tilde{a}$ containing E vertices for nonnegative integer E. From Theorem 2, it follows that for a nonnegative integer V, we have

$$\mathrm{mon}\,((m,n),p,\tilde{a},V) = \bigcup_{E\in\mathbb{Z}} \; \bigcup_{\tau\in \mathrm{pathoid}\;((m,n),p,\tilde{a},E)} \left(\Psi^{1}(\tau) \bigcap \mathrm{mon}\,((m,n),p,\tilde{a},V)\right).$$

From Lemma 4, it follows that

$$|\mathrm{mon}\,((m,n),p,\tilde{a},V)| = \sum_{E\in\mathbb{Z}} \; \sum_{\tau\in \mathrm{pathoid}\;((m,n),p,\tilde{a},E)} \begin{bmatrix} V-E \\ S+p-1+E \end{bmatrix}. \tag{3.1}$$

This gives

$$|\mathrm{mon}\,((m,n),p,\tilde{a},V)| = \sum_{E\in\mathbb{Z}} |\mathrm{pathoid}((m,n),p,\tilde{a},E)| \begin{bmatrix} V-E \\ S+p-1+E \end{bmatrix}. \tag{3.2}$$

We have

Theorem 5. *For positive integers m, n, p; and a bivector $\tilde{a}$ of length p bounded by (m, n), for nonnegative integer V,*

$$|\mathrm{mon}\,((m,n),p,\tilde{a},V)\,| = \sum_{E\in\mathbb{Z}} |\mathrm{pathoid}((m,n),p,\tilde{a},E)| \begin{bmatrix} V-E \\ S+p-1+E \end{bmatrix}.$$

When we view a twisted binomial coefficient as a usual binomial coefficient i.e.

$$\begin{bmatrix} V-E \\ S+p-1+E \end{bmatrix} = \binom{V+S+p-1}{S+p-1+E},$$

we see that $\left\{ \binom{V+S+p-1}{S+p-1+E} \right\}_{E=0}^{\infty}$ is a family of $\mathbb{Q}$-linearly independent polynomials in $\mathbb{Q}[V]$ where V is treated as an indeterminate over $\mathbb{Q}$. From (2.4) and Theorem 5, we get

Theorem 6. *For positive integers m, n, p; and a bivector $\tilde{a}$ of length p bounded by (m, n), we have for a nonnegative integer E,*

$$|pathoid((m,n),p,\tilde{a},E)| = H_E^*((m,n),p,\tilde{a}).$$

Remark: We would like to point out that a p-tuple of vertical pathoids τ of size E based on $\tilde{a}$ can be viewed as those elements of $\text{mon}((m,n),p,\tilde{a},E)$ where any two vertices of τ from the same row lie on different components of $\varphi(\tau)$. The other way to view them is as subsets of $\text{rec}(m,n)$ containing E vertices and index at most p and when restricted to the first a_i rows, its index is at most $i-1$ and when restricted to the first $b_i - 1$ columns, index is at most $i-1$ for $i = 1, 2, \ldots, p$.

Example III. Let us consider $\text{rec}(3,4)$, $p = 2$ and $\tilde{a} = (1\,2,\ 1\,3)$.

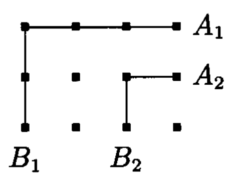

Figure 5.

We observe that our Theorem 6 tells that,

$$H_E^*((m,n),p,\tilde{a}) = \sum_{e_1+e_2=E} \begin{bmatrix} 3 \\ e_1 \end{bmatrix} \begin{bmatrix} 1 \\ e_2 \end{bmatrix} \begin{vmatrix} \binom{2}{e_1} & \binom{2}{1-e_1} \\ \binom{1}{2-e_2} & \binom{1}{1-e_2} \end{vmatrix}$$

For $E = 0$, it is clear that there is only one tuple of vertical pathoids having 0 elements and $H_0^* = 1$. For $E = 1$, we have $H_1^* = 6$ and the six tuples of vertical pathoids of size 1 are shown in Figure 6. The squared point indicates a member of the tuple of vertical pathoids based on a bivector $\tilde{a}$ having one element.

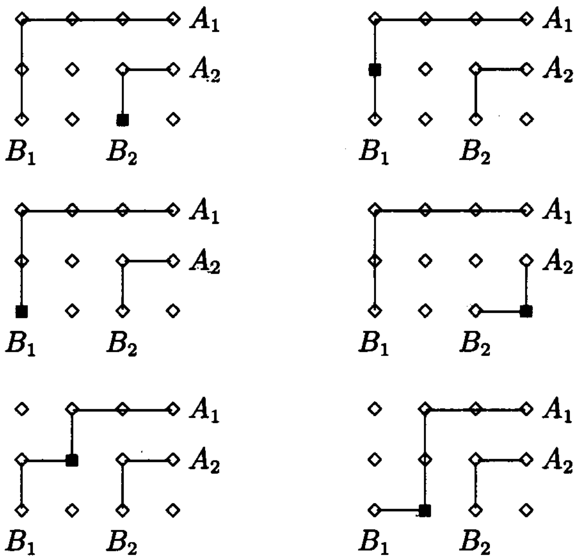

Figure 6.

For $E = 2$, $H_E^* = 12$, for $E = 3$, $H_E^* = 8$ and we have $H_E^* = 0$ for $E > 3$. Figure 7 illustrates as 12 members of pathoid $((m,n),p,\tilde{a})$ having 2 points (they are "squared" in the Figure).

From Theorem 6 and from the definition of p-tuple of vertical pathoids based on a bivector $\tilde{a}$, we have

Lemma 7. *For positive integers m, n, p; and a bivector $\tilde{a}$ of length p bounded by (m,n), we have for a nonnegative integer E,*

(i) $H_E^*((m,n),p,\tilde{a}) \geq 0$,

(ii) $\{E \in \mathbb{Z} : H_E^*((m,n),p,\tilde{a}) \neq 0\} = \{i \in \mathbb{Z} : 0 \leq i \leq R\}$ *where $\mathbb{Z}$ is the set of integers.*

It is clear that to each p-tuple of vertical pathoids of size R based on $\tilde{a}$, there corresponds a unique p-tuple of nonintersecting paths based on $\tilde{a}$. This tells us that the two sets pathoid $((m,n),p,\tilde{a},R)$ and the set of all p-tuples of nonintersecting paths based on $\tilde{a}$ in rec(m,n) i.e. path* $((m,n),p,\tilde{a})$ are equicardinal. By using a Theorem of Gessel–Viennot in [2] counting the p-tuples of nonintersecting paths in the triangular array and Cauchy–Binet Theorem on minors of the product of matrices, we can easily see that

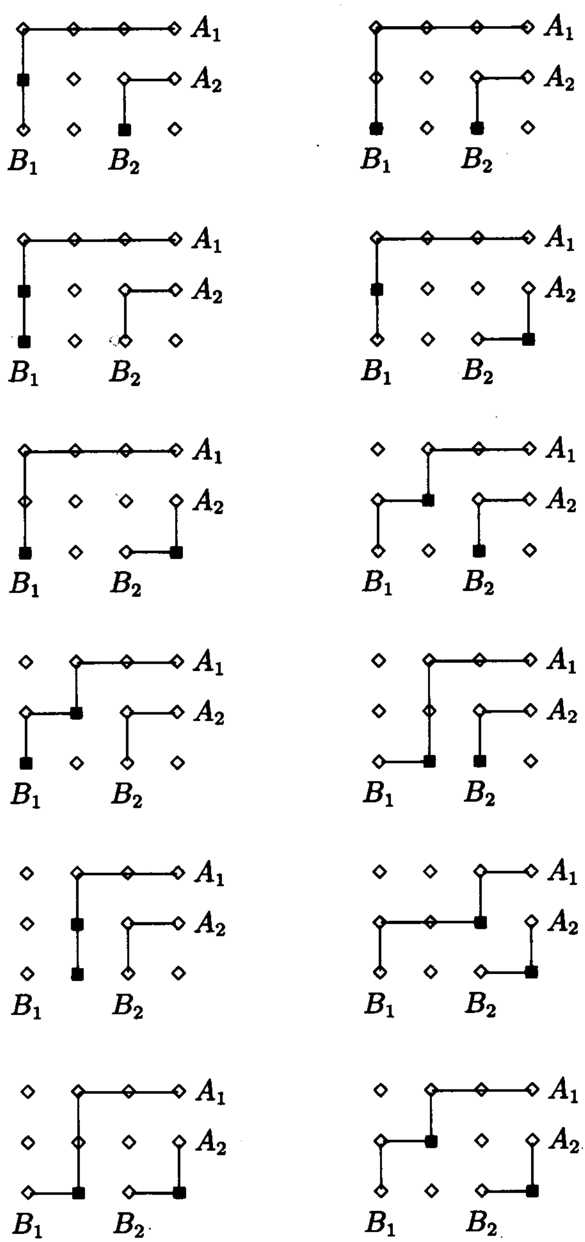

Figure 7.

$$\text{path}^{*}\left((m,n),p,\tilde{a}\right) = \det_{1\le i,\ j\le p} \begin{bmatrix} m-a_i \\ n-b_j \end{bmatrix}.$$

This gives us an interesting determinantal identity

$$H_R^{*}\left((m,n),p,\tilde{a}\right) = \sum_{e_1+e_2+\ldots+e_p=R} \left\{\prod_{i=1}^{p} \begin{bmatrix} n-b_i \\ e_i \end{bmatrix}\right\} \det_{1\le i,\ j\le p} \binom{m-a_j+j-i}{m-a_j-e_i}$$
$$= \det_{1\le i,\ j\le p} \begin{bmatrix} m-a_i \\ n-b_j \end{bmatrix}.$$

References

[1] S. Abhyankar, *Enumerative Combinatorics of Young Tableaux*, Vol. 115, Monographs in Pure and Applied Mathematics, Marcel — Dekker, 1988.

[2] I. Gessel and X. G. Viennot, Determinants, Paths and Hook Length Formulae, *Advances in Mathematics* **58**(1985), 300–321.

[3] I. Gessel and X. G. Viennot, Determinants, Paths and Plane Partitions, preprint.

[4] D. M. Kulkarni, Counting of Paths and Coefficients of Hilbert Polynomial, submitted.

[5] J. R. Stembridge, Nonintersecting Paths, Pfaffians and Plane Partitions, *Advances in Mathematics* **83**(1990), 96–131.

Devadatta M. Kulkarni

Department of Mathematics
Oakland University
Rochester, Michigan 48309 U.S.A.
and
Department of Mathematics
University of Poona
Pune 411007 India