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First Distributed Sets in the Association Scheme of Bilinear Forms

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ABSTRACT

By using the Erdős-Ko-Rado theorem for the association scheme of bilinear forms, we completely classify the first distributed sets of $H_q(n,d)$.

1. Introduction

Let $\chi = (X, \{R_i\}_{i=0}^{d})$ be a symmetric association scheme and $\mathbb{R}^X = V_0 \perp V_1 \perp \dots V_d$ be the orthogonal decomposition of $\mathbb{R}^X$, the vector space generated by the elements of X over the field of real numbers $\mathbb{R}$, with each V_i being the maximal common eigenspace of the adjacency matrices of χ. A vector $\omega \in \mathbb{R}^X$ is said to be a *general* vector if and only if $< \omega, x > \neq 0$ for all x in X. For i, $1 \leq i \leq d$, we define the *i-th distribution invariant* $Vt_i(\chi)$ as $Vt_i(\chi) = \min_\omega |\{x \in X \mid \omega \in V_i, < \omega, x > > 0 \text{ and } \omega \text{ general}\}|$. Let ω be a general vector for which $|\{x \in X \mid < \omega, x > > 0\}| = Vt_i(\chi)$. Then the set $\{x \in X \mid < \omega, x > > 0\}$ is called the ith distributed set. Several papers have appeared on distribution invariants since it was introduced in [2].

In this paper we classify all the first distributed sets in the association scheme of bilinear forms. We show that each first distributed set is an intersecting family in the sense of Erdős-Ko-Rado, and then we use Huang's work [5] to establish the classification. In the process we also show that the first distribution invariant of $H_q(n,d)$ is $q^{n(d-1)}$.

2. Association scheme of bilinear forms

Let X be the collection of all $d \times n$ matrices over the finite field $GF(q)$, where d and n are positive integers with $d \leq n$. Define $R_i = \{(y,z) \in X \times X \mid \text{rank}(y-z) = i\}$. Then $\chi = (X, \{R_i\}_{i=0}^d)$ becomes a symmetric association scheme. Indeed, it is a distance regular graph. It is called the association scheme of bilinear forms or the generalized Hamming scheme and is denoted by $H_q(n,d)$. Let $\mathbb{R}^X = V_0 \perp \ldots \perp V_d$ be its associated vector space, where $V_0 =< (1,1,\ldots,1) >$.

First we will provide a nice structure for V_1. Let H be a vector space of dimension $n+d$ over $GF(q)$ and W be a fixed subspace of H of dimension n. Let $X_i = \{A \mid A \subseteq H \text{ is an } i\text{-space and } A \cap W = \{0\}\}$. Then $(X_d, X_{d-1}, \supseteq)$ becomes a semilinear incidence structure. Any incidence structure isomorphic to $\pi = (X_d, X_{d-1}, \supseteq)$ is called an (n,q,d)-attenuated space. Ray-Chaudhuri and Sprague have shown ([9], page 6) that $H_q(n,d)$ is isomorphic to the adjacency graph of π. From now onwards we will take $H_q(n,d)$ as the adjacency graph of $(X_d, X_{d-1}, \supseteq)$.

Now let M_{1d} be the matrix whose rows and columns are indexed by the elements of X_d and X_1 respectively, and $(x,y)^{\text{th}}$ entry of M_{1d} is 1 if $x \supseteq y$ and 0 otherwise. Let U_1 be the row space of M_{1d}. Then by using long but routine calculations one can show that $U_1 \simeq V_0 \perp V_1$, where V_1 is the first eigenspace of $H_q(n,d)$. Similar calculations are done for q-analogue of the Johnson scheme in [8]. Hence from the definition of M_{1d}, we see that $V_1 \simeq \{(a_v)_{v \in X_1} \mid \sum_{v \in X_1} a_v = 0\}$. Therefore, a vector $\omega = (a_v)_{v \in X_1}$ in V_1 is a general vector if and only if $\sum_{V \in X_1 \cap A} a_v \neq 0$ for all $A \in X_d$. Since the row space U_1 contains the vector indexed by a projective point v which has $q^{n(d-1)}$ entries equal to zero, the orthogonal projection of such a vector into the space V_1 has $q^{n(d-1)}$ positive and $q^{nd} - q^{n(d-1)}$ negative entries, so that the first distribution invariant of $H_q(n,d)$ is less than or equal to $q^{n(d-1)}$.

Theorem. *The first distribution invariant of $H_q(n,d)$ is $q^{n(d-1)}$. Further $Y \subseteq H_q(n,d)$ is a first distributed set if and only if there exists a projective point v which is contained in all the elements of Y.*

Proof. Let $\omega = (a_v)_{v \in X_1}$ in V_1 be a general vector. Let $D = \{A \in X_d \mid \sum_{v \in X_1 \cap A} a_v < 0\}$

Claim. $|D| \leq q^{nd} - q^{n(d-1)}$.

Suppose $|D| > q^{nd} - q^{n(d-1)}$. Then count the number of pairs (Λ, A) where Λ is a W-complement d-spread of H and A is an element of D which is contained in Λ. The reader is referred to [7] for the definition and existence of W-complement d-spreads. We claim that D must contain a W-complement d-spread of H. Suppose not. Then for any d-dimensional subspace A contained in D, the number of such pairs (Λ, A) involving A is equal to the number of W-complement d-spreads of H containing A and this number is independent of A. Let t denote the total number of pairs (Λ, A), and let r denote the number of W-complement d-spreads of H containing A. Then

$$t = |D|r \tag{1}$$

On the otherhand, since we have assumed that there does not exist a W-complement d-spread of H consisting of d-subspaces only from D, we must have

$$t \leq (\text{ number of } W \text{ complement } d\text{-spreads of } H \text{ in total})\,(q^n - 1). \tag{2}$$

Let s = number of W-complement d-spreads of H in total. From (1) & (2) we see that $|D| \leq (s/r)(q^n - 1)$. But notice that

$$\frac{s}{r} = \frac{\text{numb. of } d\text{-dim. subspaces in } W}{\text{numb. of } d\text{-dim. subspaces in a } W\text{-complement } d\text{-spread}} = \frac{q^{nd}}{q^n}$$

Therefore, $|D| \leq q^{nd} - q^{n(d-1)}$, a contradiction. Therefore, D must contain a W-complement d-spread Ψ. This implies $0 = \sum_{v \in X_1} a_v = \sum_{A \in \Psi} \left(\sum_{v \in X_1 \cap A} a_v\right) < 0$ a contradiction again. Therefore, our assumption that $|D| > q^{nd} - q^{n(d-1)}$ is not true. i.e. $|D| \leq q^{nd} - q^{n(d-1)}$. When $|D| = q^{nd} - q^{n(d-1)}$, we must have equality in (2) which clearly implies that each W-complement d-spread must contain precisely one $A \in X_d$ not from D, i.e. $\sum_{v \in X_1 \cap A} A_v > 0$. This implies that for any two $A, B \in X_d \backslash D$, dim $(A \cap B) \geq 1$ since otherwise they would be contained in a W-complement d-spread. Thus the set of all $A \in X_d$ with $\sum_{v \in X \cap A} a_v > 0$ is an intersecting family in the sense of Erdős–Ko–Rado (see [5]). Now the last part of the theorem follows from the work of Huang [5] on the Erdős–Ko–Rado theorem for the association scheme of bilinear forms. ■

Remark. Since $H_q(n,d)$ carries the structure of an abelian group, from the definition and theorem 4 of [3] and theorem 6.3 of [4] the following can be deduced: For any t with $1 \le t \le d$, there exists a $\{1,2,\ldots,t\}$-design X of size $|X| = q^{nt}$ and consequently for the distribution numbers in the sum of the eigenspaces $V_1 \perp \ldots \perp V_t$ we have a lower bound of

$$\frac{|H_q(n,d)|}{|X|} = q^{nd-nt}\,.$$

By using a construction similar to the one preceding the statement of the theorem, we can get $Vt_{\{1,2,\ldots,t\}}\ (H_q(n,d)) = q^{n(d-t)}$.

References

[1] E. Bannai and T. Ito, *Algebraic Combinatorics I, Association Schemes,* Lecture Note Series in Math., Benjamin-Cummings New York, 1984.

[2] T. Bier, A distribution invariant for association schemes and strongly regular graphs, *Linear Algebra and Appl.* **57**(1984), 105–113.

[3] T, Bier and P. Delsarte, Some bounds for the distribution numbers of an association scheme, *Europ. J. Comb.* **9**(1988), 1–5.

[4] P. Delsarte, Bilinear forms over a finite field, with applications to coding theory, *J. of Comb. Theory* A **5**(1978), 226–241.

[5] T, Huang, An analogue of the Erdős–Ko–Rado theorem for the distance-regular graphs of bilinear forms, *Discrete Mathematics* **64**(1987), 191–198.

[6] N. Manickam, *Distribution Invariants of Association Schemes,* Ph. D. Thesis, The Ohio State University, 1986.

[7] N. Manickam, *W*-Complement *d*-spreads, Singleton systems, *Europ. J. Comb.* (1987), 437–439.

[8] N. Manickam and N. M. Singhi, 'First distribution Invariants and EKR theorems, *J. of Comb. Theory* Ser. A (1988), 91–103.

[9] D. K. Ray-Chaudhuri and A. Sprague, A combinatorial characterization of attenuated spaces, *Utilitas Math.* **15**(1979), 3–29.

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