COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
60. SETS, GRAPHS AND NUMBERS, BUDAPEST (HUNGARY), 1991

Antichain Decompositions of a Partially Ordered Set

E. C. MILNER* and M. POUZET†

1. Introduction

A basic question in the theory of ordered sets is how many chains are needed to cover a given partially ordered set. In this paper we study the corresponding question for antichains which also leads to interesting questions. The *antichain covering number* of a partially ordered set $\mathcal{P} = \langle P, \leq \rangle$ is the smallest cardinal $a(\mathcal{P}) = \kappa$ such that P is a union of κ antichains. One natural way to partition a partially ordered set into antichains is the following. Let $f : \mathcal{P} \to \mathcal{P}'$ be a strictly increasing map from $\mathcal{P}$ into the partially ordered set $\mathcal{P}' = \langle P', \leq_{\mathcal{P}'} \rangle$ (i.e. $x <_{\mathcal{P}} y \Rightarrow f(x) <_{\mathcal{P}'} f(y)$), then the set of images $\{f^{-1}(x) : x \in P'\}$, which we call the *kernel of* f is an antichain covering of P. Define the *rank* of $\mathcal{P}$, $r(\mathcal{P})$, to be the smallest cardinal κ such that there are a poset $\mathcal{P}'$ of cardinality $|P'| = \kappa$, and a strictly increasing map $f : \mathcal{P} \to \mathcal{P}'$. Since a partially ordered set has a linear extension of the same cardinality, it follows that the rank, $r(\mathcal{P})$, is also the least κ such that there is a strictly increasing map from $\mathcal{P}$ into a chain of size κ.

---

* Research supported by NSERC Grant No. A5198.

† Research partially supported by the Centre Jaques Cartier, Lyon, and PRC Mathématiques-Informatique (CNRS).

Problem 1. *Is $a(\mathcal{P}) = r(\mathcal{P})$?*

In the special case of a well founded partially ordered set $\mathcal{P}$, the ordinal height of an element $x \in P$ is inductively defined by setting

$$h(x) = \sup\{h(y) + 1 : y < x\}.$$

This is a strictly increasing map from $\mathcal{P}$ into an ordinal and the levels of $\mathcal{P}$, $L_\alpha = \{x \in P : h(x) = \alpha\}$ $(\alpha < h(\mathcal{P}))$, give an antichain decomposition. Thus for a well founded partially ordered set $\mathcal{P}$ we have that

$$a(\mathcal{P}) \leq r(\mathcal{P}) \leq |h(\mathcal{P})|, \tag{1.1}$$

where $h(\mathcal{P}) = \min\{\alpha : L_\alpha(\mathcal{P}) = \emptyset\}$ is the *height* of P. Equivalently, $h(\mathcal{P})$ is the least ordinal α such that there is a strictly increasing map from $\mathcal{P}$ into α. If $\mathcal{P}$ is well founded and of finite height, then $\mathcal{P}$ contains a chain of size $h(\mathcal{P})$ and so there is equality in (1.1). Kurepa [14] observed that $a(\mathcal{P}) \leq \aleph_0$ holds if and only if there is a strictly increasing map from $\mathcal{P}$ into the rational chain η, and so the equality $a(\mathcal{P}) = r(\mathcal{P})$ also holds in this case. In fact, Kurepa's proof gives a little more. For an ordinal α, denote by $T_2(\alpha)$ the linearly ordered set of all sequences of 0's and 1's of length α which have a last 1 (i.e. $f \in T_2(\alpha) \Leftrightarrow f : \alpha \to \{0,1\}$ and there is $\xi \in \alpha$ such that $f(\xi) = 1$ and $f(\eta) = 0$ for $\xi < \eta < \alpha$) ordered lexicographically. In particular, $T_2(\omega)$ is order isomorphic to η.

Theorem 1.1. *Let $a(\mathcal{P}) = \kappa$. Then there is a strictly increasing map from the partially ordered set $\mathcal{P}$ into the chain $T_2(\kappa)$, and hence*

$$a(\mathcal{P}) \leq r(\mathcal{P}) \leq 2^{<\kappa} \left(= \sum\{2^\rho : \rho < \kappa\}\right). \tag{1.2}$$

Proof. By definition P is a union of κ pairwise disjoint antichains $A_\nu (\nu < \kappa)$. For $x \in A_\nu$ define $f_x \in T_2(\kappa)$ by setting $f_x(\xi) = 1$ if and only if $\xi \leq \nu$ and there is $y \in A_\xi$ such that $y \leq x$. The map $x \mapsto f_x$ is strictly increasing. ■

Corollary 1.2. *GCH* $\Rightarrow$ $(\forall \mathcal{P}) a(\mathcal{P}) = r(\mathcal{P})$.

A *universal chain of cardinal* μ is a chain $\mathcal{C} = \langle C, \leq \rangle$ which embeds every chain having the same cardinality μ. There is a connection between Problem 1 and the existence of a universal chain of a certain cardinality. For, example, if μ is an infinite cardinal number and $2^{<\mu} = \mu$, then we get as an immediate corollary of Theorem 1.1 the well-known fact that $T_2(\mu)$ is a universal chain of cardinal μ. We prove the following.

Theorem 1.3. *Let μ be an infinite cardinal. If*

$$(\forall\mathcal{P})\ a(\mathcal{P}) = \mu \Rightarrow r(\mathcal{P}) = \mu,$$

then there is a universal chain of cardinal μ.

Proof. Let $\mathcal{D} = \langle D, \leq\rangle$ be the direct sum of all chains of cardinality μ, i.e. $D = \mu \times R$, where $R \subseteq 2^{\mu\times\mu}$ is the set of all linear orders on μ and $\langle\xi,\rho\rangle \leq \langle\eta,\sigma\rangle$ holds if and only if $\rho = \sigma$ and $(\xi,\eta) \in \rho$. For each $\xi \in \mu$, the set $D_\xi = \{\xi\} \times R$ is an antichain in $\mathcal{D}$, and so $a(\mathcal{D}) \leq \mu$; on the other hand $\mathcal{D}$ contains chains of cardinality μ, and so $a(\mathcal{D}) = \mu$. Therefore, by the hypothesis of the theorem $r(\mathcal{D}) = \mu$, and so there is a strictly increasing map from $\mathcal{D}$ into a chain $\mathcal{C}$ of cardinality μ; from the definition of $\mathcal{D}$ it follows that $\mathcal{C}$ is a universal chain of cardinal μ. ∎

Although the existence of a universal chain of cardinality μ does not ensure that $2^{<\mu} = \mu$ (see Kojman & Shelah [13]), we can ask the following question.

Problem 2. *Is the converse of Theorem 1.3 true?* (We are grateful to the referee for bringing to our attention a result of S. Shelah [24] (Theorem 4.7) which shows that it is consistent that ($\neg$CH+$\exists$ a universal chain of cardinality $\aleph_1$.)

We call an antichain decomposition of a partially ordered set $\mathcal{P}$ *rankable* if it is the kernel of some strictly increasing map $f : \mathcal{P} \to \mathcal{P}'$, and we consider such decompositions in §2. Another natural way to partition a partially ordered set $\mathcal{P} = \langle P, \leq\rangle$ into antichains is the following. Choose a maximal antichain $A \subseteq P$. Then $P\backslash A = P_0 \cup P_1$, where $P_0 = \{x \in P : x < a$ for some $a \in A\}$ and $P_1 = \{x \in P : a < x$ for some $a \in A\}$. Now repeat this on P_0 and P_1, choosing maximal antichains $A_0 \subseteq P_0$, $A_1 \subseteq P_1$ and defining sets P_{00}, P_{01}, P_{10}, P_{11} etc. The process is continued until P is exhausted. We call such a decomposition a *tree antichain decomposition* of $\mathcal{P}$, and we discuss these decompositions in §3 in connection with the special partially ordered set $S(\lambda,\kappa)$ defined below.

As usual, κ^+ denotes the successor cardinal of κ, and $[\lambda]^\kappa = \{X \subseteq \lambda : |X| = \kappa\}$. We shall denote by $\mathcal{S}(\lambda,\kappa)$ the set $[\lambda]^\kappa$ ordered by inclusion. In §3 we will prove the following theorem; this result was also obtained independently by M. Scheepers [21].

Theorem 1.4. *For infinite cardinals λ, κ there is a strictly increasing map from $\mathcal{S}(\lambda,\kappa)$ into $\mathcal{S}(\kappa^+,\kappa)$.*

An immediate corollary is the following.

Corollary 1.5. $a(\mathcal{S}(\lambda,\kappa)) \leq r(\mathcal{S}(\lambda,\kappa)) \leq 2^\kappa$ *if* $\omega \leq \kappa \leq \lambda$.

The inequality $a(\mathcal{S}(\lambda,\kappa)) \leq 2^\kappa$ is attributed to Milner & Erdős in [6], but no proof was ever published. Independent proofs have been obtained by F. Galvin [10], M. Scheepers [21] and Z. Szentmiklóssy (unpublished) [25]. The proof we give of Theorem 1.4 uses an idea of Galvin. Note that, for $\lambda > \kappa$ the equality $a(\mathcal{S}(\lambda,\kappa)) = r(\mathcal{S}(\lambda,\kappa)) = 2^\kappa$ follows if there is a $\subseteq$-increasing chain of cardinality 2^κ in $[\lambda]^\kappa$, or equivalently, in $[\kappa^+]^\kappa$. But this assertion is known to be independent of the axioms of set theory (see W. Mitchell [17]).

Another easy corollary of Theorem 1.4 is the following. For an element x of the partially ordered set $\mathcal{P} = \langle P, \leq\rangle$, denote by $\mathcal{P}(\leq x)$ the set $\{y \in P : y \leq x\}$ ($\mathcal{P}(< x)$, $\mathcal{P}(> x)$ etc. are similarly defined).

Corollary 1.6. *If $\mathcal{P} = \langle P, \leq\rangle$ is a partially ordered set, κ an infinite cardinal and $|P(\leq x)| < \kappa$ for all $x \in P$, then $r(\mathcal{P}) \leq 2^{<\kappa}$.*

Proof. By hypothesis $P = \bigcup\{P_\mu : \mu < \kappa\}$, where $P_\mu = \{x \in P : |P(\leq x)| = \mu\}$. Since the map $x \mapsto |P(\leq x)|$ is (weakly) increasing, it follows that $r(\mathcal{P}) \leq \sum\{r(\mathcal{P}_\mu) : \mu < \kappa\}$, where $\mathcal{P}_\mu = \mathcal{P}|P_\mu$. Also, the map $x \mapsto P(\leq x)$ is a strictly increasing map from $\mathcal{P}_\mu$ into $\langle [P]^\mu, \subseteq\rangle$, and so, by Corollary 1.5, $r(\mathcal{P}_\mu) \leq |\mathcal{S}(\mu^+,\mu)| = 2^\mu$. Thus $r(\mathcal{P}) \leq 2^{<\kappa}$. ■

It is perhaps of some interest to note that Corollary 1.6 also follows directly from Theorem 1.1 and a set-mapping theorem of Fodor [8] which states that: *if κ is an infinite cardinal and $f : E \to \wp(E)$ is a map satisfying $x \notin f(x)$ and $|f(x)| < \kappa (\forall x \in E)$, then E is a union of κ free sets $H_\rho (\rho < \kappa)$* (the set H is *free* if $f(x) \cap H = \emptyset$ for all $x \in H$). Indeed, the free sets for the set-mapping defined by $f(x) = P(< x)$ are antichains. Thus, if $|P(\leq x)| < \kappa$ for all $x \in P$, then $a(\mathcal{P}) \leq \kappa$ by Fodor's theorem and so $r(\mathcal{P}) \leq 2^{<\kappa}$ by Theorem 1.1.

Problem 3. *By the definition of rank, if f is a strictly increasing map from $\mathcal{P}$ into $\mathcal{P}'$, then $r(P(\leq x)) \leq |P'(\leq f(x))|$ $(\forall x \in P)$. Is there necessarily a partially ordered set $\mathcal{P}'$ and a strictly increasing map $f : \mathcal{P} \to \mathcal{P}'$ such that equality holds for each $x \in P$?*

If the answer to Problem 3 is positive, then the hypothesis $|P(\leq x)| < \kappa$ in Corollary 1.6 can be replaced by the weaker assumption that, for all $x \in P$, $r(P(\leq x)) < \kappa$. However, we do not know the answer to the following question.

Problem 4. *Is there a cardinal $\varphi(\kappa)(\leq 2^{<\kappa}?)$ such that, if $r(P(\leq x)) < \kappa$ holds for all $x \in P$, then $r(\mathcal{P}) \leq \varphi(\kappa)$?*

The *chain-number* of a poset $\mathcal{P}$, $c(\mathcal{P})$, is the smallest cardinal κ such that $|C| \leq \kappa$ for every chain C in $\mathcal{P}$; this number is *attained* if $c(\mathcal{P}) = |C|$ for some chain C in $\mathcal{P}$. Clearly $c(\mathcal{P}) \leq a(\mathcal{P})$. There is equality if $c(\mathcal{P})$ is finite, but in general $a(\mathcal{P})$ cannot be bounded by any function of $c(\mathcal{P})$. Indeed, this can be seen by a slight modification of a well known example of Perles [18]. The direct product $\kappa \otimes \kappa$, of the infinite cardinal κ with itself (i.e. the ordering of $\kappa \times \kappa$ in which $(\alpha, \beta) \leq (\alpha', \beta') \Leftrightarrow \alpha \leq \alpha'$ and $\beta \leq \beta'$) is easily seen to have no infinite antichain and is not a union of fewer than κ chains. This example shows that there is no immediate generalization of Dilworth's theorem [4] to the case of a partially ordered set whose antichains are not finitely bounded in size. For our purposes we consider instead the strict product, $\mathcal{P} = \kappa \odot \kappa^*$, of the cardinal κ and its dual κ^* (i.e. $\mathcal{P} = (\kappa \times \kappa, \leq)$, where $(\alpha, \beta) < (\alpha', \beta') \Leftrightarrow \alpha < \alpha'$ and $\beta > \beta'$). Since two elements are comparable in $\mathcal{P}$ if and only if they are incomparable in $\kappa \otimes \kappa$, it follows that $\mathcal{P}$ contains no infinite chain, $c(\mathcal{P}) = \omega$, but $\mathcal{P}$ is not a union of fewer than κ antichains and so $a(\mathcal{P})(= r(\mathcal{P})) = \kappa$.

In a similar way, the height of a well founded partially ordered set is not bounded by any function of its rank. Indeed, consider the partially ordered set $\mathcal{P}_\alpha$ of height α in which every chain is finite constructed by transfinite induction in the following way (see [11]). Let $\mathcal{P}_1$ be the one-element poset. For limit α, $\mathcal{P}_\alpha$ is the direct sum of the $\mathcal{P}_\beta$ for all $\beta < \alpha$, and for a successor ordinal $\alpha = \beta + 1$, $\mathcal{P}_\alpha$ is obtained by adding one new element to $\mathcal{P}_\beta$ which is a maximum. By induction it follows that $\mathcal{P}_\alpha$ contains no infinite chain, $h(\mathcal{P}_\alpha) = \alpha$, and $r(\mathcal{P}_\alpha) \leq r(\mathcal{P}_\alpha^*) \leq h(\mathcal{P}_\alpha^*) \leq \omega$, where $\mathcal{P}^*$ is the dual of $\mathcal{P}$.

If $\mathcal{G} = (V, E)$ is an undirected graph a subset $A \subseteq V$ is a *clique* of $\mathcal{G}$ if $[A]^2 \subseteq E$, and A is an *independent subset* if $[A]^2 \cap E = \emptyset$. The *chromatic number* of $\mathcal{G}$, $\chi(\mathcal{G})$, is the least cardinal κ such that V is a union of κ independent sets. We also define the *clique number* of $\mathcal{G}$, $\psi(\mathcal{G})$, to be the smallest cardinal λ such that $|A| \leq \lambda$ for every clique A in $\mathcal{G}$. Clearly, $\psi(\mathcal{G}) \leq \chi(\mathcal{G})$, but, in general, $\psi(\mathcal{G})$ is not necessarily attained and there may be strict inequality. A graph $\mathcal{G}$ is *good* if it contains a clique of cardinality

$\chi(\mathcal{G})$ (in which case $\psi(\mathcal{G})$ is attained), and $\mathcal{G}$ is *perfect* if every induced subgraph is good. If $\mathcal{G}$ is the comparability graph of a partially ordered set $\mathcal{P}$ (i.e. $x, y \in P$ are joined by an edge in $\mathcal{G}$ if and only if $x < y$ or $y < x$), then the independent sets are antichains and the cliques are chains of $\mathcal{P}$; thus $a(\mathcal{P}) = \chi(\mathcal{G})$ and $c(\mathcal{P}) = \psi(\mathcal{G})$, and the chain number of $\mathcal{P}$ is attained if the comparability graph is good. Corollary 1.8 below gives a necessary and sufficient condition for the comparability graph of a partially ordered set having countable antichain number to be perfect.

Denote by $\mathcal{C}_n$ the chain of length n and let $\Omega = \bigoplus\{\mathcal{C}_n : n < \omega\}$ be the direct sum of all the finite chains $\mathcal{C}_n$ $(n < \omega)$; also denote by K_Ω the comparability graph of Ω. We will prove (see §4) the following characterisation for a graph of countable chromatic number to be perfect.

Theorem 1.7. *If $\mathcal{G}$ is a graph with chromatic number $\chi(\mathcal{G}) \leq \omega$, then $\mathcal{G}$ is perfect if and only if every finite subgraph is perfect and $\mathcal{G}$ contains no induced subgraph isomorphic to K_Ω.*

In particular, since the comparability graph of a finite partially ordered set is perfect, we have the following corollary. We say that the partially ordered set $\mathcal{P} =< P, \leq>$ *embeds* the partially ordered set $\mathcal{P}' =< P', \leq'>$, and write $\mathcal{P}' \leq \mathcal{P}$, if and only if there is a strictly increasing map (*an embedding*) $f : \mathcal{P}' \to \mathcal{P}$ such that $x \leq' y \Leftrightarrow f(x) \leq f(y)$.

Corollary 1.8. *If $\mathcal{P}$ is a partially ordered set such that $a(\mathcal{P})$ is countable, then the comparability graph of $\mathcal{P}$ is perfect if and only if $\mathcal{P}$ does not embed Ω.*

The condition that $\chi(\mathcal{G})$ be countable in Theorem 1.7 may not be essential, but some condition on $\mathcal{G}$ is needed. The *interval order* on a linearly ordered set $\mathcal{S}$ is the ordering on the set $\mathcal{I}(\mathcal{S})$ of all non-trivial open intervals of $\mathcal{S}$ of the form (u, v) with $u < v$ in $\mathcal{S}$, ordered so that $I < J$ holds if $x < y$ holds in $\mathcal{S}$ for all $x \in I$ and $y \in J$. Now suppose that $\mathcal{S}$ is a Suslin line, i.e. a chain of cardinality $\aleph_1$ which contains no countable dense subset and contains no uncountable family of pairwise disjoint open intervals. The interval order on $\mathcal{S}$, $\mathcal{I}(\mathcal{S})$, contains no uncountable chain and is not the union of countably many antichains. (For, if $\mathcal{F} = \{(x_\alpha, y_\alpha) : \alpha < \lambda\}$ is any family of pairwise intersecting members of $\mathcal{I}(\mathcal{S})$, then the set of left endpoints $\{x_\alpha : \alpha < \lambda\}$ contains a countable cofinal subset and hence there is a countable set which intersects all the members of $\mathcal{F}$. Consequently, $\mathcal{I}(\mathcal{S})$ is not a union of countably many antichains.) Thus, the comparability graph of $\mathcal{I}(\mathcal{S})$ is not good. On the other hand, the interval order $\mathcal{I}(\mathcal{S})$ of any linearly ordered set $\mathcal{S}$ does not embed Ω; in fact it does not even embed $\mathcal{C}_2 \oplus \mathcal{C}_2$ the direct sum of two 2-element chains.

By Theorem 1.1, if the partially ordered set $\mathcal{P}$ embeds η, the rational chain, and if $a(\mathcal{P}) = \omega$, then $c(\mathcal{P})$ is an attained supremum. In §5 we consider this question, whether $c(\mathcal{P})$ is an attained supremum, for an arbitrary scattered partially ordered set ($\mathcal{P}$ is *scattered* if it does not embed η). We will prove the following result.

Theorem 1.9. *For a scattered partially ordered set $\mathcal{P}$ the following statements are equivalent:*

(i) $\mathcal{P}$ it does not embed Ω;

(ii) $\mathcal{P}'$ contains a chain of cardinality $r(\mathcal{P}')$ whenever $\mathcal{P}' \leq \mathcal{P}$;

(iii) The comparability graph of $\mathcal{P}$ is perfect.

Moreover, if $\mathcal{P}$ is well founded then these conditions are also equivalent to

(iv) $\mathcal{P}'$ contains a chain of order type $h(\mathcal{P}')$ whenever $\mathcal{P}' \leq \mathcal{P}$.

Let us say that the poset $\mathcal{P}$ has property **AF** if there is a strictly increasing, surjective map from $\mathcal{P}$ onto some chain such that the image of any antichain of $\mathcal{P}$ is finite. Write $\mathcal{P} \in \mathbf{AF}(C)$ if the chain C is a witness that P has the property **AF**. Note that if $\mathcal{P}$ has the property **AF**, then it does satisfy *(i)-(iii)* of Theorem 1.9. In fact, we have the following lemma.

Lemma 1.10. *Let $\mathcal{P} \in \mathbf{AF}(C)$.*

(i) If C is infinite, then $\mathcal{P}$ contains a chain of cardinality $|C|$;

(ii) the comparability graph of $\mathcal{P}$ is perfect and hence P does not embed Ω;

(iii) if $\mathcal{P}$ is scattered (well founded), then so also is C.

Proof. Let f be a surjective map from $\mathcal{P}$ onto C such that the image of any antichain is finite. For each $y \in C$ let $x_y \in P$ be such that $f(x_y) = y$. Since $f|P'$ is a bijective map from $P' = \{x_y : y \in C\}$ onto C, it follows that P' contains no infinite antichain.

(i) By the Erdős-Dushnik-Miller theorem [5], P' contains a chain D of cardinality $|D| = |C|$. It follows that $|D| \leq a(\mathcal{P}) \leq r(\mathcal{P}) \leq |C|$ and so $a(\mathcal{P}) = r(\mathcal{P}) = |D|$.

(ii) If C is finite we can assume that $|C|$ is minimal, in which case $\mathcal{P}$ contains a chain of size $|C|$. Thus, in any case, $\mathcal{P}$ contains a chain of cardinality

$a(\mathcal{P}) = r(\mathcal{P})$. Since the property **AF** is hereditary, it follows that the comparability graph of $\mathcal{P}$ is perfect and therefore does not embed Ω.

(iii) C is order isomorphic to a linear extension of $\mathcal{P}'$, and it is known (see e.g. [2]) that any linear extension of a scattered (resp. well founded) poset with no infinite antichain is also scattered (resp. well founded). ∎

As a corollary of this and the last part of Theorem 1.9 we obtain the following result of Milner & Sauer [16]; the special case when f is the height function was obtained independently by Pouzet [19] and also, for countable $\mathcal{P}$, by Schmidt [22].

Corollary 1.11. *If there is a strictly increasing ordinal-valued function, f, on a poset $\mathcal{P}$ such that the image of every antichain is finite, then $\mathcal{P}$ contains a chain of order type $h(\mathcal{P})$.*

Proof. Since $\mathcal{P}$ has property **AF** its comparability graph is perfect and so $\mathcal{P}$ does not embed Ω. ∎

Note that the last part of Theorem 1.9 is actually stronger than Corollary 1.11 since, as the following examples show, the fact that a poset does not embed Ω does not ensure that it has property **AF**.

Examples: The interval order of any linearly ordered set does not embed Ω. We will show that the interval orders $\mathcal{I}(\eta)$ and $\mathcal{I}(\omega_1)$ do not have the property **AF**.

Consider first $\mathcal{I}(\eta)$. Suppose $f : \mathcal{I}(\eta) \to \mathcal{P}'$ is an order preserving map such that the image of any antichain is finite, and let $\mathcal{A}$ be the kernel of f. Since $\mathcal{A}$ is countable, there is some real x such that $\{x\} \neq \bigcap\{\overline{I} : I \in A\}$ for all $A \in \mathcal{A}$, where $\overline{I}$ denotes the ordinary closure of I in $\mathbf{R}$. Now consider the antichain $B = \{I \in \mathcal{I}(\eta) : x \in I\}$. Since B is closed under finite intersections, and $f[B]$ is finite, there is some $B' \subseteq B$ which is coinitial in $\langle B, \subseteq\rangle$ and is such that f is constant on B', say $f(I) = y$ for all $I \in B'$. Let $A = f^{-1}(y)$. Then $B' \subseteq A \in \mathcal{A}$. But then $\{x\} = \bigcap\{\overline{I} : I \in A\}$, and this is a contradiction.

Now suppose that $f : \mathcal{I}(\omega_1) \to \mathcal{P}'$ is an order preserving map such that every antichain has a finite image. In particular then, the image of $A(\alpha) = \{I \in \mathcal{I}(\omega_1) : \alpha \in I\}$, $f[A(\alpha)]$, is finite. Hence there is an uncountable set $W \subseteq \omega_1$ such that the sets $f[A(\alpha)]$ $(\alpha \in W)$ form a Δ-system, i.e. $f[A(\alpha)] \cap f[A(\beta)] = D$ whenever α, β are distinct members of W. Let $|D| = k < \omega$. There are ordinals $\alpha_0 < \alpha_1 < \ldots < \alpha_{2k+1}$ in W and

$k+1$ intervals belonging to $\mathcal{I}(\omega_1)$, $I_0 < \ldots < I_k$, such that $\{\alpha_{2i}, \alpha_{2i+1}\} \subseteq I_i$ $(0 \leq i \leq k)$. Now $f(I_i) \in f[A(\alpha_{2i})] \cap f[A(\alpha_{2i+1})] \subseteq D$. But f is strictly increasing and so $f(I_0) < f(I_1) < \ldots < f(I_k)$, and this contradicts the fact that $|D| = k$.

The above examples leave open the following question.

Problem 5. *If $\mathcal{P}$ is a countable, scattered poset which does not embed Ω, does $\mathcal{P}$ have property* **AF***?*

Another question with a similar theme is the following.

Problem 6. *If $\mathcal{P}$ is scattered and does not embed Ω, is there a finite number of chains $C_1, \ldots, C_n$ such that a chain C embeds into $\mathcal{P}$ if and only if it embeds into some C_i?*

2. Rankable antichain decompositions.

An *antichain decomposition* of a partially ordered set $\mathcal{P} =< P, \leq >$ *is* a set $\mathcal{A}$ of pairwise disjoint non-empty antichains whose union $\cup\mathcal{A} = P$. We say that $\mathcal{A}$ is *rankable* if it is the kernel of some strictly increasing map f from $\mathcal{P}$ into some partially ordered set $\mathcal{P}' = \langle P', \leq \rangle$, i.e. $\mathcal{A} = \{f^{-1}(y) : y \in P'\} \backslash \{\emptyset\}$. We denote by $\mathcal{L}(\mathcal{P})$ the lattice of all partitions of P ordered by refinement. The rankable antichain decomposition of $\mathcal{P}$ is *maximal* (in $\mathcal{L}$) if it is not a proper refinement of some other rankable antichain decomposition. For any antichain decomposition $\mathcal{A}$ of $\mathcal{P}$ we define a binary relation $\mathcal{R}(\mathcal{A}) = \{(A, A') : \exists x \in A \exists x' \in A' (x \leq x')\}$ on $\mathcal{A}$; also we denote by $\hat{\mathcal{R}}(\mathcal{A})$ the transitive closure of $\mathcal{R}(\mathcal{A})$.

Theorem 2.1. *Let $\mathcal{P} = \langle P, \leq \rangle$ be a partially ordered set.*

1. *An antichain decomposition $\mathcal{A}$ of $\mathcal{P}$ is rankable if and only if $\mathcal{R}(\mathcal{A})$ is acyclic.*
2. *(i) A rankable antichain decomposition of $\mathcal{P}$ is a refinement of some maximal rankable antichain decomposition.*

   (ii) A rankable antichain decomposition $\mathcal{A}$ is maximal if and only if $\hat{\mathcal{R}}(\mathcal{A})$ is a linear order.

Proof. (1) If $\mathcal{R}(\mathcal{A})$ is acyclic, then its transitive closure $\hat{\mathcal{R}}(\mathcal{A})$ is an ordering of $\mathcal{A}$ and the natural map f from $\mathcal{P}$ to $\langle \mathcal{A}, \hat{\mathcal{R}}(\mathcal{A}) \rangle$, defined by $x \in f(x)$, is strictly increasing. Since $\mathcal{A}$ is the kernel of f, it follows that $\mathcal{A}$ is rankable.

Conversely, if $\mathcal{A}$ is the kernel of some strictly increasing map f from $\mathcal{P}$ to some partially ordered set $\langle Q, \leq_Q \rangle$, then the relation

$$\mathcal{R}' = \{(A, A') : \exists x \in A \in \mathcal{A} \exists x' \in A' \in \mathcal{A}(f(x) \leq_Q f(x'))\}$$

is a partial ordering of $\mathcal{A}$ which includes $\mathcal{R}(\mathcal{A})$. Hence $\mathcal{R}(\mathcal{A})$ is acyclic.

(2) Let **A** be a directed family of rankable antichain decompositions of $\mathcal{P}$, i.e. whenever $\mathcal{A}_1, \mathcal{A}_2 \in \mathbf{A}$, then there is $\mathcal{A} \in \mathbf{A}$ such that $\mathcal{A}_i \leq \mathcal{A}$ $(i = 1,2)$ in $\mathcal{L}(\mathcal{P})$. Then $\mathcal{B} = \sup(\mathbf{A})$ (in $L(\mathcal{P})$) is a rankable antichain decomposition of $\mathcal{P}$. This follows from the fact that, for any finite set $F \subseteq P$, there is some $\mathcal{A} \in \mathbf{A}$ such that $\mathcal{A} \cap F = \mathcal{B} \cap F$, where $\mathcal{A} \cap F$ is the partition $\{A \cap F : A \in \mathcal{A}\} \backslash \{\emptyset\}$ of F. Since each $\mathcal{A} \in \mathbf{A}$ is rankable, it follows by (1) that $\mathcal{R}(\mathcal{A})$ is acyclic and hence that $\mathcal{R}(\mathcal{B})$ is also acyclic. Thus, again by (1), $\mathcal{B}$ is a rankable antichain decomposition of $\mathcal{P}$. The first part of (2) now follows by Zorn's lemma.

Now suppose that $\hat{\mathcal{R}}(\mathcal{A})$ is a linear order and that $\mathcal{A}$ is a refinement of the rankable antichain decomposition $\mathcal{A}'$. The natural map f from $\langle \mathcal{A}, \hat{\mathcal{R}}(\mathcal{A}) \rangle$ to $\langle \mathcal{A}', \hat{\mathcal{R}}(\mathcal{A}') \rangle$ defined by $A \subseteq f(A)$ $(A \in \mathcal{A})$ is strictly increasing. Since $\hat{\mathcal{R}}(\mathcal{A})$ is a linear ordering it follows that f must be 1-1 and hence the identity map. Thus $\mathcal{A}$ is a maximal rankable antichain decomposition. Conversely, suppose $\hat{\mathcal{R}}(\mathcal{A})$ is not a linear ordering of $\mathcal{A}$. Then there are $A_1, A_2 \in \mathcal{A}$ which are incomparable in $\hat{\mathcal{R}}(\mathcal{A})$. Consider the antichain decomposition $\mathcal{A}'$ obtained from $\mathcal{A}$ by replacing A_1 and A_2 by $A_1 \cup A_2$. If $\mathcal{R}(\mathcal{A}')$ contains a cycle, then it must be of the form $\langle A_1 \cup A_2, B_1, \ldots, B_n \rangle$. Then there are x_0, y_0 in $A_1 \cup A_2$, and x_i, y_i in B_i $(1 \leq i \leq n)$ such that $x_i < y_{i+1}$ for $i \leq n$, where $y_{n+1} = y_0$. Since $\mathcal{R}(\mathcal{A})$ is acyclic it follows that x_0, y_0 do not both belong to A_1 or A_2, and therefore (A_1, A_2) or (A_2, A_1) belongs to $\hat{\mathcal{R}}(\mathcal{A})$, and this is a contradiction. Therefore, $\mathcal{R}(\mathcal{A}')$ is acyclic and $\mathcal{A}'$ is rankable, and so $\mathcal{A}$ is not maximal. This proves the second part of (2). ∎

We digress somewhat here in order to discuss a related idea. (For example, the necessity of the condition in (2)(ii) of Theorem 2.1 follows immediately from Corollary 2.5 below.)

First we recall the so-called amalgamation lemma for partially ordered sets (see Fraïssé [9], p.34).

Lemma 2.2. *If $\mathcal{P}_1 =< P_1, \leq_1 >$ and $\mathcal{P}_2 = \langle P_2, \leq_2 \rangle$ are partially ordered sets such that $\leq_1$ and $\leq_2$ agree on $P = P_1 \cap P_2$, and if $\leq$ is the transitive* *closure of* $\leq_1 \cup \leq_2$*, then the identity maps from* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *into* $\langle P_1 \cup P_2, \leq \rangle$ *are embeddings.*

In the category whose objects are partially ordered sets and morphisms are the strictly increasing maps, this simply says: *if* $f_1 : \mathcal{P} \to \mathcal{P}_1$ *and* $f_2 : \mathcal{P} \to \mathcal{P}_2$ *are embeddings, then there are a partially ordered set* $\mathcal{P}'$ *and embeddings* $g_1 : \mathcal{P}_1 \to \mathcal{P}'$, $g_2 : \mathcal{P}_2 \to \mathcal{P}'$ *such that* $f_1 \circ g_1 = f_2 \circ g_2$*; in other words* diagram 2.1 *commutes.*

$$\begin{array}{ccccc} & & f_1 & & \\ & \mathcal{P} & \Longrightarrow & \mathcal{P}_1 & \\ f_2 & \Downarrow & & \Downarrow & g_1 \\ & \mathcal{P}_2 & \Longrightarrow & \mathcal{P}' & \\ & & g_2 & & \end{array}$$

Diagram 2.1

We shall prove the following related result.

Lemma 2.3. Transferability lemma: *Let* $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$ *be partially ordered sets and suppose that* $f_1 : \mathcal{P} \to \mathcal{P}_1$ *is a strictly increasing map, and* $f_2 : \mathcal{P} \to \mathcal{P}_2$ *is an embedding. Then there are a partially ordered set* $\mathcal{P}'$*, an embedding* $g_1 : \mathcal{P}_1 \to \mathcal{P}'$*, and a strictly increasing map* $g_2 : \mathcal{P}_2 \to \mathcal{P}'$ *such that* $f_1 \circ g_1 = f_2 \circ g_2$*, i.e.* diagram 2.2 *commutes.*

$$\begin{array}{ccccc} & & f_1 & & \\ & \mathcal{P} & \longrightarrow & \mathcal{P}_1 & \\ f_2 & \Downarrow & & \Downarrow & g_1 \\ & \mathcal{P}_2 & \longrightarrow & \mathcal{P}' & \\ & & g_2 & & \end{array}$$

Diagram 2.2

Proof. Let $\mathcal{A}_1$ be the kernel of f_1 and consider the relation

$$\mathcal{R}_1 = \{(A, A') : \exists x \in A \in \mathcal{A}_1 \exists x' \in A' \in \mathcal{A}_1 (f_1(x) \leq_1 f_1(x'))\}$$

on $\mathcal{A}_1$. This includes $\mathcal{R}(\mathcal{A}_1)$ since f_1 is strictly increasing. Clearly $\mathcal{R}_1$ is acyclic and so $\hat{\mathcal{R}}_1$, its transitive closure, is an ordering of $\mathcal{A}_1$, and the map $\bar{f}_1$, defined by $\bar{f}_1(A) = f_1(a)$ for $a \in A$, is an embedding of $\langle \mathcal{A}_1, \hat{\mathcal{R}}_1 \rangle$ into $\mathcal{P}_1$. Also the map $h_1 : \mathcal{P} \to \langle \mathcal{A}_1, \mathcal{R}_1 \rangle$, defined by $x \in h_1(x)$, is strictly increasing and $f_1 = h_1 \circ \bar{f}_1$. Without loss of generality we can assume that $P \subseteq P_2$ and that f_2 is the identity map i_P. Let $\mathcal{A}_2 = \mathcal{A}_1 \cup \{\{x\} : x \in P_2 \backslash P\}$ be the finest antichain partition of P_2 containing $\mathcal{A}_1$. The relation $\mathcal{R}_2 = \mathcal{R}(\mathcal{A}_2) \cup \mathcal{R}_1$ is acyclic. (Note that if $A, A' \in \mathcal{A}_1$ and $x \in P_2 \backslash P$ and $(A, \{x\}) \in \mathcal{R}_2$, $(\{x\}, A') \in \mathcal{R}_2$, then $(A, A') \in \mathcal{R}_1$; since $\mathcal{R}_1$ and $\mathcal{P}_2$ are both acyclic, it follows that a cycle in $\mathcal{R}_2$ contains exacly one member $A \in \mathcal{A}_1$; but this is impossible since A is an antichain in $\mathcal{P}_2$.) Thus $\hat{\mathcal{R}}_2$, the transitive closure of $\mathcal{R}_2$, is an ordering and the identity map is an embedding of $\langle \mathcal{A}_1, \hat{\mathcal{R}}_1 \rangle$ into the partially ordered set $\langle \mathcal{A}_2, \hat{\mathcal{R}}_2 \rangle$. By the amalgamation property it follows that there are $\mathcal{P}'$ and embeddings $g_1 : \mathcal{P}_1 \to \mathcal{P}'$ and $k_2 : \langle \mathcal{A}_2, \hat{\mathcal{R}}_2 \rangle \to \mathcal{P}'$ such that $\bar{f}_1 \circ g_1 = i_{\mathcal{A}_1} \circ k_2$. The natural map $h_2 : \mathcal{P}_2 \to \langle \mathcal{A}_2, \hat{\mathcal{R}}_2 \rangle$ given by $x \in h_2(x)$ is strictly increasing and hence so also is $g_2 = h_2 \circ k_2 : \mathcal{P}_2 \to \mathcal{P}'$. Clearly $h_1 \circ i_{\mathcal{A}_1} = f_2 \circ h_2 (= i_P \circ h_2)$. Thus

$$f_1 \circ g_1 = h_1 \circ \bar{f}_1 \circ g_1 = h_1 \circ i_{\mathcal{A}_1} \circ k_2 = f_2 \circ h_2 \circ k_2 = f_2 \circ g_2.$$

This argument is shown diagramatically in diagram 2.3.

$$\begin{array}{ccccccc} & & h_1 & & \bar{f}_1 & & \\ & \mathcal{P} & \longrightarrow & \langle \mathcal{A}_1, \hat{\mathcal{R}}_1 \rangle & \Longrightarrow & \mathcal{P}_1 & \\ f_2 & \Downarrow & & \Downarrow i_{\mathcal{A}_1} & & \Downarrow & g_1 \\ & \mathcal{P}_2 & \longrightarrow & \langle \mathcal{A}_2, \hat{\mathcal{R}}_2 \rangle & \Longrightarrow & \mathcal{P}' & \\ & & h_2 & & k_2 & & \end{array}$$

Diagram 2.3

Remark 1. We used the amalgamation property to derive the transferability lemma. The converse is also possible. Suppose $f_1 : \mathcal{P} \to \mathcal{P}_1$ and $f_2 : \mathcal{P} \to \mathcal{P}_2$ are embeddings. By the transferability lemma there are partially ordered sets $\mathcal{T}_1$, $\mathcal{T}_2$, embeddings $k_1 : \mathcal{P}_1 \to \mathcal{T}_1$, $k_2 : \mathcal{P}_2 \to \mathcal{T}_2$ and strictly increasing maps $h_1 : \mathcal{P}_1 \to \mathcal{T}_2$, $h_2 : \mathcal{P}_2 \to \mathcal{T}_1$ such that $f_1 \circ k_1 = f_2 \circ h_2$ and $f_1 \circ h_1 = f_2 \circ k_2$. Consider the strict product $\mathcal{P}' = \mathcal{T}_1 \odot \mathcal{T}_2$ and the maps $g_1 : \mathcal{P}_1 \to \mathcal{P}'$, $g_2 : \mathcal{P}_2 \to \mathcal{P}'$ given by $g_1(y) = (k_1(y), h_1(y))(y \in P_1)$, $g_2(z) = (h_2(z), k_2(z))(z \in P_2)$. It is a simple matter to check that g_1 and g_2 are embeddings, and $f_1 \circ g_1 = f_2 \circ g_2$.

An immediate consequence of the transferability lemma (in fact an equivalent form) is the following.

Corollary 2.4. *If $\mathcal{P}$ is a subposet of $\mathcal{T}$, then any rankable antichain decomposition $\mathcal{A}$ of $\mathcal{P}$ can be extended to a rankable antichain decomposition of $\mathcal{T}$.*

Proof. Apply the lemma with $\mathcal{P}_1 = \langle \mathcal{A}, \hat{\mathcal{R}}(\mathcal{A}) \rangle$ and $\mathcal{P}_2 = \mathcal{T}$. ■

We deduce from this another corollary.

Corollary 2.5. *1. Two distinct elements x, y of a partially ordered set $\mathcal{P}$ are incomparable if and only if they belong to the same class in some rankable antichain decomposition of $\mathcal{P}$.*

2. If f is a strictly increasing map from $\mathcal{P}$ into $\mathcal{P}'$, then the image $f[P]$ of $\mathcal{P}$ is a linearly ordered subset of $\mathcal{P}'$ if and only if, whenever g is a strictly increasing map from $\mathcal{P}'$ into some poset $\mathcal{Q}$, then the restriction $g|f[P]$ is injective.

Proof. (1) follows immediately from Corollary 2.4. For (2), if x, y are incomparable elements of $f[P]$ (in $\mathcal{P}'$), then by (1) there is a strictly increasing map g from $\mathcal{P}'$ to some partially ordered set $\mathcal{Q}$ which identifies x and y. ■

Remark 2. The necessity of the condition in (2)(*ii*) of Theorem 2.1 follows from Corollary 2.5 (*2*). For, if $\mathcal{A}$ is rankable antichain decomposition of $\mathcal{P}$, then the natural map f from $\mathcal{P}$ into $\mathcal{P}' = \langle \mathcal{A}, \hat{\mathcal{R}}(\mathcal{A})\rangle$ for which $x \in f(x)$ for all $x \in P$, is strictly increasing; also, by the maximality of $\mathcal{A}$, any strictly increasing map from $\mathcal{P}'$ to some poset $\mathcal{Q}$ must be injective. Hence $\langle \mathcal{A}, \hat{\mathcal{R}}(\mathcal{A})\rangle$ is a linear ordering.

The ideas behind the transferability lemma lead to a third proof of Corollary 1.7 and to the theorem of Fodor [8] referred to in §1.

For an oriented graph $\mathcal{G} = (V, E)$ and an element $x \in V$, we write $E(\leftarrow x) = \{y \in V : (y, x) \in E\}$. A subset $A \subseteq V$ is *closed* if $E(\leftarrow x) \subseteq A$ for all $x \in A$. Let $\mathbf{G}$ be a class of oriented graphs and let κ be a cardinal number. We say that the graph $\mathcal{G}_0 = (V_0, E_0)$ has the $(\mathbf{G}, \kappa)$-*extension property* if, whenever $\mathcal{G} = (V, E) \in \mathbf{G}$, $V' \in [V]^{<\kappa}$, $x \in V \backslash V'$, and there is an edge-preserving map f from $\mathcal{G}|V'$ to $\mathcal{G}_0$, then there is an edge-preserving map f_1 from $\mathcal{G}|V' \cup \{x\}$ to $\mathcal{G}_0$ which extends f (thus f extends to an edge-preserving map of $\mathcal{G}|V_1$ to $\mathcal{G}_0$ for any $V_1 \supseteq V'$ of cardinality $\leq \kappa$).

Lemma 2.6. *Suppose that $\mathcal{G}_0$ has the $(\mathbf{G}, \kappa)$-extension property and that $\mathcal{G} = (V, E) \in \mathbf{G}$ is such that $|E(\leftarrow x)| < \kappa$ for all $x \in V$. Then, whenever Q is a closed subset of V and f is an edge-preserving map from $\mathcal{G}|Q$ into $\mathcal{G}_0$, then there is an edge-preserving map from $\mathcal{G}$ into $\mathcal{G}_0$ which extends f.*

Proof. Consider the collection, $\mathcal{F}$, of all edge-preserving maps g from some subgraph $\mathcal{G}|R$ to $\mathcal{G}_0$, where R is a closed subset of $\mathcal{G}$ and g extends f. The set $\mathcal{F}$, ordered by inclusion, has a maximal member h. It suffices to show that the domain of h, say H, is equal to V. For a contradiction, suppose there is some $x \in V \setminus H$. The minimal closed set $K = \{k_\alpha : \alpha < \lambda\}$ containing x has cardinality $\lambda \leq \kappa$. We inductively define an increasing sequence of edge-preserving maps $h_\alpha (\alpha \leq \lambda)$ from induced subgraphs of $\mathcal{G}$ into $\mathcal{G}_0$ which extend h as follows. Put $h_0 = h$. For limit α, set $h_\alpha = \cup\{h_\beta : \beta < \alpha\}$. For $\alpha = \beta + 1$ a successor, since $X_\beta = E(\leftarrow x_\beta) \cap \text{domain}(h_\beta)$ has cardinality less than κ, there is an edge-preserving extension of $h_\beta | X_\beta$ to $X_\beta \cup \{x_\beta\}$, say h'_β and we put $h_\alpha = h_\beta \cup h'_\beta$. This defines the h_α for $\alpha \leq \lambda$, and h_λ is an edge-preserving extension of h to the closed set $H \cup K$, and this contradicts the maximality of h. ■

Remark 3. The set-mapping result of Fodor [8] mentioned in §1 is a special case of Lemma 2.6. Clearly the complete (loop-free) graph $K_\kappa = (\kappa, D)$, where $D = \{(\alpha, \beta) \in \kappa \times \kappa : \alpha \neq \beta\}$, has the $(\mathbf{G}, \kappa)$-extension property, where $\mathbf{G}$ is the class of all loop-free directed graphs. If f is a set-mapping on S such that $|f(x)| < \kappa$ for all x in S, apply the lemma with $Q = \emptyset$ and $\mathcal{G} = (S, E)$ where $E = \{(y, x) : y \in f(x)\}$. There is an edge preserving map, g, from $\mathcal{G}$ to K_κ and S is the union of the f-free sets $g^{-1}(\alpha)(\alpha < \kappa)$.

Remark 4. Corollary 1.6 also follows from Lemma 2.6. Let $\mathcal{P} = \langle P, \leq \rangle$ be a partially ordered set such that $|\mathcal{P}(\leq x)| < \kappa$, and let $\mathcal{G} = \langle P, F \rangle$ be the comparability graph of $\mathcal{P}$ (i.e. $F = \{(x, y) : x < y\}$). If $\kappa = \aleph_\alpha$ is regular, then $T_2(\kappa)$ defined in §1 is an η_α-set (see Hausdorff [11]), i.e. has the property that whenever A, B are subsets of cardinality $< \kappa$ and every element of A precedes every element of B, then there is some element c such that $a < c < b$ holds for all $a \in A$, $b \in B$. Therefore, the comparability graph of $T_2(\kappa)$ has the $(\mathbf{G}, \kappa)$-extension property, where $\mathbf{G}$ is the class of all directed graphs $\mathcal{H} = (V, E)$ with the property that $|E(\leftarrow x)| < \kappa$ for all $x \in V$. Since $\mathcal{G} \in \mathbf{G}$ it follows from Lemma 2.6 that there is a strictly increasing map from $\mathcal{P}$ into $T_2(\kappa)$. Suppose κ is singular and that $\kappa_\xi (\xi < \text{cf}(\kappa))$ is an increasing sequence of regular cardinals cofinal in κ. Let $\mathcal{P}_\xi$ be the sub-poset of $\mathcal{P}$ induced by the set $P_\xi = \{x \in P : |\mathcal{P}(\leq x)| < \kappa_\xi\}$. Then we can define strictly increasing maps $f_\xi : \mathcal{P}_\xi \to T_2(\kappa_\xi)$ for $\xi < \text{cf}(\kappa)$ such that $f_{\xi+1}$ extends f_ξ and $f_\xi = \cup\{f_\eta : \eta < \xi\}$ for limit ξ. Then $f = \cup f_\xi$ is a strictly increasing map from $\mathcal{P}$ into $T_2(\kappa)$.

An immediate corollary of Theorem 2.1 is the following result of Bonnet & Pouzet [2].

Theorem 2.7. *For a partially ordered set $\mathcal{P}$, the inequality $r(\mathcal{P}) \leq \kappa$ holds if and only if there is a strictly increasing map f from $\mathcal{P}$ onto a chain $\mathcal{C}$ of* *size $\leq \kappa$ such that the order on $\mathcal{C}$ is the transitive closure of the image under f of the ordering on $\mathcal{P}$, i.e. whenever $f(x) < f(y)$ then there are $n < \omega$ and a sequence $x = x_0, y_0, x_1, y_1, \ldots, x_n, y_n = y$ in $\mathcal{P}$ such that $f(x_i) = f(y_i)$ $(i \leq n)$ and $y_i < x_{i+1}$ $(i < n)$.*

We also have the following related result.

Theorem 2.8. *If κ is an infinite cardinal number, then the inequality $r(\mathcal{P}) \leq \kappa$ holds if and only if there is an embedding of the partially ordered set $\mathcal{P}$ into a strict product of chains each of cardinality at most κ.*

Proof. The sufficiency is obvious, for, if $\mathcal{P}'$ is the strict product of a family of partially ordered sets $\mathcal{P}_i (i \in I)$, then $r(\mathcal{P}') \leq \min\{r(\mathcal{P}_i) : i \in I\}$ (since each projection map is strictly increasing).

For the necessity we first show that $r(\mathcal{P}) \leq \kappa$ implies the following separation property (*): *if $x \not\leq y$ in $\mathcal{P}$, then there are a chain $\mathcal{C} = \mathcal{C}_{x,y}$ and a strictly increasing map $f = f_{x,y} : \mathcal{P} \to \mathcal{C}$ such that $|C| \leq \kappa$ and $f(y) < f(x)$.*

Clearly (*) holds if $y < x$. Suppose that x and y are incomparable. Divide P into three parts, $P_0 = \{z \in P : z < x \text{ or } z < y\}$, $P_1 = \{x, y\}$ and $P_2 = P \backslash (P_0 \cup P_1)$. Since $r(\mathcal{P}) \leq \kappa$, there are chains $\mathcal{C}_0$, $\mathcal{C}_2$ having cardinalities at most κ, and strictly increasing maps f_0, f_2 from $\mathcal{P}|P_0$ and $\mathcal{P}|P_2$ into $\mathcal{C}_0$, $\mathcal{C}_2$ respectively. Let $\mathcal{C}_1$ be the two-element chain $\{x, y\}$ with $y < x$, and let f_1 be the identity map on $\{x, y\}$. Since κ is infinite, (*) holds with $\mathcal{C}$ the lexicographic sum $\mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2$ and $f = f_0 \cup f_1 \cup f_2$.

If $\mathcal{P}$ is a chain the theorem is obvious. Assume that $\mathcal{P}$ is not a chain and consider the strict product $\mathcal{P}'$ of all the chains $\mathcal{C}_{x,y}$ with $x \not\leq y$ in $\mathcal{P}$. The function $f : \mathcal{P} \to \mathcal{P}'$, where $f(z) = (f_{x,y}(z) : x \not\leq y)$ is an embedding. ■

Remark 5. Theorem 2.8 fails for finite κ. For example, the partially ordered set Λ (on three elements a, b, c in which the only order relations are $a < b$ and $c < b$) has rank 2 but Λ cannot be embedded into a strict product of two-element chains.

Remark 6. The rank of a strict product of partially ordered sets may be strictly less than the rank of each of its components. This follows from a result of Shelah [23] (see also Todorčević [26]) who proved that there is a Countryman-type, i.e. there is a chain $\mathcal{C}$ of cardinality ω_1 such that the ordinary direct product $\mathcal{C} \otimes \mathcal{C}$ is a union of countably many chains. Since there is an exact correspondence between a chain in $\mathcal{C} \otimes \mathcal{C}$ and an antichain in the strict product $\mathcal{P} = \mathcal{C} \odot \mathcal{C}^*$, where $\mathcal{C}^*$ is the reverse of $\mathcal{C}$, it follows that $a(\mathcal{P}) = \omega$. Therefore, by Theorem 1.1, $r(\mathcal{P}) = \omega$ also. But $r(\mathcal{C}) = r(\mathcal{C}^*) = \omega_1$.

Remark 7. We end this section with one additional remark about strict products of chains. The *dimension* of a partially ordered set $\mathcal{P} =< P, \leq>$, as defined by Dushnik & Miller [5], is the least cardinal κ such that $\leq$ *is* equal to the intersection of κ total orders on P. It is well-known that this is equal to the smallest cardinal κ such that $\mathcal{P}$ can be embedded into the direct product of κ chains (Hiraguchi [12]). Although not so well known, it can easily be shown (see [15]) that the dimension of $\mathcal{P}$ is also equal to the smallest cardinal κ such that $\mathcal{P}$ can be embedded into the strict product of κ chains.

3. Tree-antichain decompositions.

There is another very simple and natural way to partition a partially ordered set $\mathcal{P} = \langle P, \leq \rangle$ into antichains. Choose a maximal antichain $A \subseteq P$. Then $P \backslash A = P_0 \cup P_1$, where $P_0 = \{x \in P : \exists a \in A(x < a)\}$ and $P_1 = \{x \in P : \exists a \in A(a < x)\}$. Now repeat this on P_0 and P_1, choosing maximal antichains $A_0 \subseteq P_0$, $A_1 \subseteq P_1$ and defining sets P_{00}, P_{01}, P_{10}, P_{11} etc. The process is continued until there are no elements left.

We extend our earlier notation by writing $\mathcal{P}(< X)$ ($\mathcal{P}(> X)$ etc.), where $X \subseteq P$, to denote the set $\{y \in P : \exists x \in X(y < x)\}$; thus $\mathcal{P}(< x) = \mathcal{P}(< \{x\})$. Also, if $f \in 2^\alpha$ is a sequence of 0's and 1's of length α, then we denote by $f * \epsilon$ the sequence $f \cup \{(\alpha, \epsilon)\}$ of length $\alpha + 1$. The decomposition described above can be more precisely defined as follows. Put $P_\emptyset = P$. If $f \in 2^\alpha$ is a sequence and $P_f \subseteq P$ has been defined, then let A_f be a maximal antichain in $\mathcal{P}|P_f$ and define $P_{f*0} = \mathcal{P}(< A_f) \cap P_f$, $P_{f*1} = \mathcal{P}(> A_f) \cap P_f$. For limit α and $f \in 2^\alpha$, let $P_f = \cap\{P_{f|\beta} : \beta < \alpha\}$. This defines P_f and A_f for all ordinal sequences of 0's and 1's.

Lemma 3.1. *There is an ordinal μ such that $|\mu| \leq |P|$ and $P = \cup\{A_f : f \in 2^{<\mu}\}$.*

Proof. Observe that, by the construction, if f, g are two ordinal sequences of 0's and 1's, then (i) $f \subset g$ (i.e. g is a strict extension of f) $\Rightarrow P_g \subseteq P_f \backslash A_f$, and (ii) f, g incomparable $\Rightarrow P_f \cap P_g = \emptyset$. Thus the members of the set $\mathcal{S} \subseteq \wp(P)$ consisting of all the P_f's are pairwise either disjoint or comparable. Moreover, for any non-empty member Q of $\mathcal{S}$, there is exactly one f such that $P_f = Q$; and if $f \neq g$ then A_f and A_g are disjoint. It follows that, for $x \in P$, $\mathcal{T}_x = \{f : x \in P_f\}$ is a set and its members are pairwise comparable so that $h = \bigcup \mathcal{T}_x$ is an ordinal sequence of 0's and 1's. Clearly $x \in P_h$, so that h is the largest member of $\mathcal{T}_x$. Since $h * 0$ and $h * 1$ do not belong to $\mathcal{T}_x$, it follows that $x \in A_h$. Thus $\mathcal{A} = \{A_f : P_f \neq \emptyset\}$ is an antichain decomposition of $\mathcal{P}$. For each $A_f \in \mathcal{A}$ the sequence of sets $\langle P_{f|\alpha} : \alpha \in \text{dom } f\rangle$ is strictly decreasing, and so $|\text{dom } f| \leq |P|$ and the lemma follows. ■

The set $\mathcal{F}$, of all the sequences f of 0's and 1's such that $P_f \neq \emptyset$ described in the above construction, has the property that, if $f \in \mathcal{F}$, $\alpha \in \text{dom } f$, then $f|\alpha \in \mathcal{F}$ and $P_f \subseteq P_{f|\alpha}$. By the lemma, there is an ordinal $\mu < |P|^+$ such that $\mathcal{F} \subseteq 2^{<\mu}$ and μ is the height of the tree $\langle \mathcal{F}, \subseteq\rangle$. We call the family of sets $\mathcal{S} = \{P_f : f \in \mathcal{F}\}$ a *tree decomposition* of $\mathcal{P}$, and the corresponding family of antichains $\mathcal{A} = \{A_f : f \in \mathcal{F}\}$ a *tree antichain decomposition* of $\mathcal{P}$ with *code* $\mathcal{F}$. Note that $\langle \mathcal{S}, \supseteq\rangle$ is also a tree with root P which is order isomorphic to $\langle \mathcal{F}, \subseteq\rangle$. In fact, it is easily seen that a family, $\mathcal{S}$, of non-empty subsets of P is a tree decomposition of $\mathcal{P}$ (with a unique code) if and only if the following conditions are satisfied:

1. $\mathcal{T} = \langle \mathcal{S}, \supseteq\rangle$ is a tree with root P;
2. $Q \in \mathcal{S}$ is the least upper bound of a chain $\mathcal{C}$ in $\mathcal{T}$ if and only if $Q = \bigcap \mathcal{C}$;
3. $Q \in \mathcal{S}$ is a terminal node of $\mathcal{T}$ if and only if Q is an antichain of $\mathcal{P}$;
4. any non-terminal node Q of $\mathcal{T}$ has at most two successors in $\mathcal{T}$, say Q_1 and Q_2, where $\{Q_1, Q_2\} = \{Q \cap \mathcal{P}(< A), Q \cap \mathcal{P}(> A)\} \setminus \{\emptyset\}$ and $A = Q \setminus (Q_1 \cup Q_2)$ is a maximal antichain in $\mathcal{P}|Q$.

Let $\mathcal{A} = \{A_f : f \in \mathcal{F}\}$ be a tree antichain decomposition of $\mathcal{P}$ with code $\mathcal{F} \subseteq 2^{<\mu}$. Define a linear order $\prec$ on $\mathcal{F}$ by the rule that $f \prec g$ if and only if either (i) $f \subset g$ and $g(\text{dom } f) = 1$, or (ii) $g \subset f$ and $f(\text{dom } g) = 0$, or (iii) there is $\delta < \mu$ such that $f|\delta = g|\delta$ and $f(\delta) = 0$, $g(\delta) = 1$. (In other words, $f \prec g$ holds if and only if $f * 1$ lexicographically precedes $g * 1$.) If (i) holds, then $A_g \subseteq P_g \subseteq \mathcal{P}(> A_f)$; if (ii) holds, then $A_f \subseteq P_f \subseteq \mathcal{P}(< A_g)$; if (iii) holds, then $A_f \subseteq \mathcal{P}(< A_h)$ and $A_g \subseteq \mathcal{P}(> A_h)$, where $h = f|\delta$. Thus the natural map $F : \mathcal{P} \to \langle \mathcal{F}, \prec\rangle$ given by

$$F(x) = f \text{ if and only if } x \in A_f,$$

is strictly increasing and has kernel $\mathcal{A}$. Moreover, if $f \prec g$, then either $\langle A_f, A_g \rangle \in \mathcal{R}(\mathcal{A})$ or there is $h \in \mathcal{F}$ such that both $\langle A_f, A_h \rangle$ and $\langle A_h, A_g \rangle$ belong to $\mathcal{R}(\mathcal{A})$ and so $\hat{\mathcal{R}}(\mathcal{A})$ is a linear ordering of $\mathcal{A}$. Therefore, by Theorem 2.1, we have the following result.

Theorem 3.2. *A tree antichain decomposition of a partially ordered set is a maximal rankable antichain decomposition.*

Problem 7. *Is there a tree antichain decomposition $\mathcal{A}$ of a partially ordered set $\mathcal{P}$ such that $|\mathcal{A}| = r(\mathcal{P})$?*

Problem 7 is only of interest in the case when the rank is infinite. For, if $r(\mathcal{P})$ is finite, $\mathcal{P}$ is well founded and the levels of $\mathcal{P}$ give a tree antichain decomposition of size $r(\mathcal{P})$. In fact, in the case of finite rank there is a bound on the size of any tree antichain decomposition.

Theorem 3.3. *If $\mathcal{P}$ is a partially ordered set with finite rank $r(\mathcal{P})$, then any tree antichain decomposition of $\mathcal{P}$ has cardinality at most $2^{r(\mathcal{P})} - 1$.*

Proof. By induction on $r = r(\mathcal{P})$. If $r \leq 1$, the result is clear. Suppose that $r > 1$. Let $\mathcal{S} = \{P_f : f \in \mathcal{F}\}$ be a tree decomposition and $\mathcal{A} = \{A_f : f \in \mathcal{F}\}$ the corresponding tree antichain decomposition of $\mathcal{P}$. Since r is finite $\mathcal{P}$ is well founded and has height $h(\mathcal{P}) = r$. Any chain in P_ϵ ($\epsilon = 0$ or 1) has an extension in $A_\emptyset$ and so $r(\mathcal{P}|P_\epsilon) \leq r - 1$. Therefore, by the induction hypothesis, since $\mathcal{A}_\epsilon = \{A_f : A_f \subseteq P_\epsilon\}$ is a tree antichain decomposition of $\mathcal{P}|P_\epsilon$, we have that $|\mathcal{A}| \leq |\mathcal{A}_0| + |\mathcal{A}_1| + 1 \leq 2^r - 1$. ■

The the bound given by Theorem 3.3 is best possible. To see this consider the partially ordered set $\mathcal{P}^r$ of rank $r < \omega$ defined inductively as follows. $\mathcal{P}^1$ is the 1-element chain. Suppose that $r \geq 1$ and that $\mathcal{P}^r$ has been defined. Then $\mathcal{P}^{r+1}$ is the partially ordered set which contains two disjoint isomorphic copies of $\mathcal{P}^r$, say $\mathcal{Q}_\epsilon = \langle Q_\epsilon, \leq \rangle (\epsilon = 0, 1)$, and two additional points a, b such that $a < x$ for all $x \in Q_0$ and $y < b$ for all $y \in Q_1$ (see diagram 3.1).

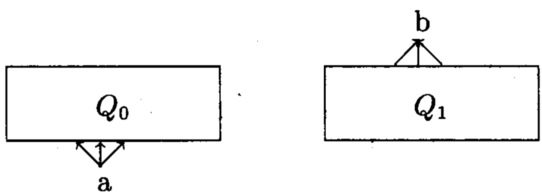

Diagram 3.1.

Clearly, by induction, $\mathcal{P}^{r+1}$ has rank $r+1$. Also, if $\mathcal{P}^r$ has a tree antichain decomposition of size $2^r - 1$, then, starting with the maximal antichain $A_\emptyset = \{a, b\}$, it follows that $\mathcal{P}^{r+1}$ has a tree antichain decomposition of size $2(2^r - 1) + 1$. Thus Theorem 3.3 is best possible since, for any $r < \omega$, $\mathcal{P}^r$ has rank r and has a tree antichain decomposition of size $2^r - 1$.

The following lemma gives a necessary and sufficient condition in order that every tree antichain decomposition of a poset be well founded.

Lemma 3.4. *For a partially ordered set $\mathcal{P}$ the following statements are equivalent:*

(i) $\mathcal{P}$ is well founded and does not embed Ω;

(ii) every tree antichain decomposition of $\mathcal{P}$ is well founded.

Proof. Suppose (ii) is false. Then there is a tree antichain decomposition $\mathcal{A} = \{A_f : f \in \mathcal{F}\}$ of $\mathcal{P}$ with code $\mathcal{F}$ such that $\langle \mathcal{F}, \prec \rangle$ embeds ω^*. By assumption, there are $f_n \in \mathcal{F}$ such that $f_0 \succ f_1 \succ f_2 \succ \ldots$. We can assume that the domains of the f_n are strictly increasing. Let g_n be the greatest lower bound of the set of elements $\{f_m : m \geq n\}$ in the tree $\langle \mathcal{F}, \subseteq \rangle$. Note that the g_n are not eventually constant. Indeed, let $\alpha_n = \operatorname{dom} g_n$. If $f_k(\alpha_n) = 1$ for all $k > n$, then $g_{n+1} \supseteq g_n{}^* 1$; if $f_m(\alpha_n) = 0$ for some $m > n$, then $g_m \supseteq g_n{}^* 0$. Therefore, we may assume that $g_0 \subset g_1 \subset g_2 \subset \ldots$ and $g_{n+1}(\alpha_n) = 0$ (since $g_n \succ g_{n+1}$). By the construction of a tree antichain decomposition it follows that $A_{g_{n+1}} \subset \mathcal{P}(< A_{g_n})$, and so for any $y \in A_{g_{n+1}}$ there is $x \in A_{g_n}$ such that $y < x$.

Now consider the partially ordered set $\mathcal{P}' = \mathcal{P}|A$, where $A = \cup\{A_{g_n} : n < \omega\}$. Clearly $\mathcal{P}'$ does not embed ω but contains chains of arbitrary finite length. We may assume that $\mathcal{P}$ does not embed Ω so that by Corollary 1.8 the comparability graph of $\mathcal{P}'$ is perfect and so $\mathcal{P}'$ contains an infinite chain. Therefore $\mathcal{P}'$, and hence also $\mathcal{P}$, embeds ω^*.

It remains to show that $(ii) \implies (i)$. Since both ω^* and Ω have tree antichain decompositions which are not well founded, it follows by Lemma 3.6 below that $\mathcal{P}$ also has such a decomposition. ■

In connection with Problem 7 and Theorem 3.3 we make two additional remarks.

Remark 1. If $\mathcal{A} = \{A_\alpha : \alpha \in \lambda\}$ is an antichain decomposition of the partially ordered set $\mathcal{P}$ obtained simply by choosing A_α to be a maximal antichain in $P \setminus \bigcup\{A_\beta : \beta < \alpha\}$, then the number of steps, $|\lambda|$, needed to exhaust P cannot be bounded by a function of $r(\mathcal{P})$ even in the case when this is finite. For, consider the partially ordered set $\mathcal{P}$ on the set $\{\{\alpha\} : \alpha < \mu\} \cup \{\mu\backslash\{\alpha\} : \alpha < \mu\}$ ordered by inclusion, where μ is any infinite cardinal number. In this case $\mathcal{P}$ has height and rank 2. However, the antichain decomposition $\mathcal{A} = \{A_\alpha : \alpha < \mu\}$ has size μ, where $A_\alpha = \{\{\alpha\}, \mu\backslash\{\alpha\}\}$ is a maximal antichain of $\mathcal{P}$.

Remark 2. In contrast to Theorem 3.3, if the rank is infinite, then the size of a tree antichain decomposition is not bounded by any function of $r(\mathcal{P})$. For example, let $\mathcal{P} = \mu \odot \mathbf{Z}$ be the strict product of the infinite cardinal μ and the chain $\mathbf{Z}$ of all integers ordered in the natural way. In this case the rank $r(\mathcal{P}) = \omega$. However, $\mathcal{A} = \{A_\alpha : \alpha < \mu\}$ is a tree antichain decomposition of $\mathcal{P}$ of cardinality μ, where $A_\alpha = \{(\alpha, n) : n \in \mathbf{Z}\}(\alpha < \mu)$.

If λ is an infinite cardinal, then $\mathcal{A} = \{[\lambda]^n : n < \omega\}$ is a tree antichain decomposition of $\mathcal{S}(\lambda, < \omega) = \langle [\lambda]^{<\omega}, \subseteq \rangle$, and so $r(\mathcal{S}(\lambda, < \omega)) = |\mathcal{A}| = \omega$. Also, if $\omega \leq \kappa \leq \lambda$, then $r(\mathcal{S}(\lambda, \kappa)) \leq 2^\kappa$ by Corollary 1.5. However, as the next theorem shows, there are tree antichain decompositions of these partially ordered sets of size at least λ.

Theorem 3.5. *Let λ, κ be infinite cardinals, $\lambda \geq \kappa$. Then (i) there is a tree antichain decomposition of $\mathcal{S}(\lambda, < \omega)$ of size λ, and (ii) there is a tree antichain decomposition of $\mathcal{S}(\lambda, \kappa)$ of size $\geq \lambda$.*

For the proof of Theorem 3.5 we need the following simple fact.

Lemma 3.6. *Let $\mathcal{P} = \langle P, \leq \rangle$ be a partially ordered set, $P' \subseteq P$. If $\mathcal{A}' = \{A'_f : f \in \mathcal{F}'\}$ is a tree antichain decomposition of $\mathcal{P}|P'$ with code $\mathcal{F}'$, then there is a tree antichain decomposition $\mathcal{A} = \{A_f : f \in \mathcal{F}\}$ of $\mathcal{P}$ with code $\mathcal{F} \supseteq \mathcal{F}'$ such that $A'_f = A_f \cap P'$ for all $f \in \mathcal{F}'$.*

Proof. We construct the tree antichain decomposition of $\mathcal{P}$ in the same way as described at the beginning of this section, but with the additional requirement that, whenever P_f is defined and $f \in \mathcal{F}'$, then $P_f \supseteq P'_f$ (where $\{P'_f : f \in \mathcal{F}'\}$ is the tree decomposition of $\mathcal{P}|P'$ associated with $\mathcal{A}'$), and then we choose A_f to be a maximal antichain in P_f which extends A'_f. Setting $P_{f*0} = P_f \cap \mathcal{P}(< A_f)$ and $P_{f*1} = P_f \cap \mathcal{P}(> A_f)$, we see that $P'_{f*\epsilon} \subseteq P_{f*\epsilon}$ if $f * \epsilon \in \mathcal{F}'$, and so the construction may be continued. The tree antichain decomposition of $\mathcal{P}$ constructed in this way has the desired property. ■

Proof of Theorem 3.5. Since $\mathcal{S}(\lambda, \kappa)$ embeds $\mathcal{S}(\lambda, < \omega)$ it follows from the lemma that it is enough to construct a partially ordered set $\mathcal{P} = \langle P, \leq \rangle$ which has a tree antichain decomposition of size λ and which can be embedded in $\mathcal{S}(\lambda, < \omega)$.

Let $F_\alpha(\alpha < \lambda)$ be a 1-1 enumeration of all the finite subsets of λ, and for $\alpha < \lambda$ put $P_\alpha = \{(\alpha, F_\beta) : F_\alpha \subseteq F_\beta\}$, $P = \cup\{P_\alpha : \alpha < \lambda\}$ and consider the ordering in which

$$(\alpha, F_\beta) < (\gamma, F_\delta) \Leftrightarrow \alpha > \gamma \text{ and } F_\beta \subset F_\delta$$

(where $\subset$ denotes strict inclusion).

If $\gamma < \alpha < \lambda$ and $(\alpha, F_\beta) \in P_\alpha$, and if $\delta < \lambda$ is chosen such that $F_\gamma \cup F_\beta \subset F_\delta$, then $(\alpha, F_\beta) < (\gamma, F_\delta)$. This shows that P_γ is a maximal antichain in $\cup\{P_\alpha : \gamma \leq \alpha < \lambda\}$, and so $\mathcal{A} = \{P_\alpha : \alpha < \lambda\}$ is a tree antichain decomposition of $\mathcal{P}$ of size λ.

We first show that $\mathcal{P}(< x)$ is finite for any $x \in P$. Indeed, suppose for a contradiction that $x = (\alpha, F_\beta)$ and $\mathcal{P}(< x)$ is infinite. Then there are $F \subset F_\beta$ and infinitely many distinct $\alpha_n > \alpha$ such that $(\alpha_n, F) \in \mathcal{P}(< x)$; but this is impossible since $(\alpha_n, F) \in P$ implies that $F_{\alpha_n} \subseteq F$. Thus $\mathcal{P}(< x)$ is finite for all $x \in P$ and hence well-founded. Let $P_\alpha(\alpha < \mu)$ denote the levels of $\mathcal{P}$ and let f_0 be any 1-1 map from P_0 into λ. Now suppose that $\alpha 0$ and that we have already defined embeddings f_β from $\mathcal{P}|\cup\{P_\gamma : \gamma \leq \beta\}$ into $\mathcal{S}(\lambda, < \omega)$ for $\beta < \alpha$ so that $f_\gamma \subset f_\beta(\gamma < \beta < \alpha)$. Then, for $x \in P_\alpha$, define $g_\alpha(x) = \{f_\beta(y) : y < x,\, y \in P_\beta\}$ and $f_\alpha(x) = g_\alpha(x) \cup \cup\{f_\beta : \beta < \alpha\}$. Clearly, $f = \cup\{f_\alpha : \alpha < \mu\}$ is an embedding of $\mathcal{P}$ into $\mathcal{S}(\lambda, < \omega)$. ■

We conclude this section with a proof of Theorem 1.4; the main idea in the following proof follows a suggestion by F. Galvin [10].

Proof of Theorem 1.4. For ordinals ξ and α we respectively denote by $S^*(\xi, \alpha)$, $S^*(\xi, < \alpha)$, and $S^*(\xi, \leq \alpha)$ the set of all subsets $X \subseteq \xi$ such that the order type of X, with the induced natural ordering of ξ, is repectively $= \alpha$, $< \alpha$ *and* $\leq \alpha$. Also we shall denote by $\mathcal{S}^*(\xi, \alpha)$ the set $S^*(\xi, \alpha)$ ordered by inclusion etc..

We will prove by induction on $\alpha < \kappa^+$ that there is a strictly increasing map f_α from $\mathcal{S}^*(\lambda, \alpha)$ into $\mathcal{S}(\kappa, \kappa)$. The theorem follows from this and the fact that $S(\lambda, \kappa)$ is the disjoint union of the sets $S^*(\lambda, \alpha)(\kappa \leq \alpha < \kappa^+)$. For, assuming the f_α exist, the function f defined by $f(X) = (\kappa \times \alpha) \cup (f_\alpha(X) \times \{\alpha\})$, where $X \in S(\lambda, \kappa)$ and α is the unique ordinal such that $X \in S^*(\lambda, \alpha)$, is a strictly increasing map from $\mathcal{S}(\lambda, \kappa)$ into $\langle [\kappa \times \kappa^+]^\kappa, \subseteq \rangle$ is isomorphic to $\mathcal{S}(\kappa^+, \kappa)$.

Obviously f_0 exists. Assume that $0 < \alpha < \kappa^+$ and that f_β has been defined for $\beta < \alpha$. Note that the induction hypothesis implies that there is a strictly increasing map, say $f_{<\alpha}$, from $\mathcal{S}^*(\lambda, < \alpha)$ into $\mathcal{S}(\kappa, \kappa)$. This follows from the fact that the function F defined by $F(X) = (\kappa \times \beta) \cup (f_\beta(X) \times \{\beta\})$ for $X \in S^*(\lambda, \beta)$ and $\beta < \alpha$, is a strictly increasing map from $\mathcal{S}^*(\lambda, < \alpha)$ into $\langle [\kappa \times \alpha]^\kappa, \subseteq \rangle$(which is isomorphic $\mathcal{S}(\kappa, \kappa)$).

For $\xi < \lambda$, let $R(\xi, \alpha) = \{X \in S^*(\xi, \alpha) : X$ is cofinal in $\xi\}$, and let $\mathcal{R}(\xi, \alpha)$ denote the corresponding ordered set ordered by inclusion. Since a well ordered set is not order isomorphic to a proper initial segment of itself, it follows that the members of $R(\xi, \alpha)$ and $R(\eta, \alpha)$ are $\subseteq$-incomparable if $\xi < \eta < \lambda$. Therefore, since $S^*(\lambda, \alpha)$ is the disjoint union of the sets $R(\xi, \alpha)$ $(\xi < \lambda)$, it will be enough to show that, for each $\xi < \lambda$, there is a strictly increasing map $f_{\xi,\alpha}$ from $\mathcal{R}(\xi, \alpha)$ into $\mathcal{S}(\kappa, \kappa)$.

Case 1. $\alpha = \beta + 1$ is a successor ordinal. If $R(\xi, \alpha)$ is non-empty, then $\xi = \eta + 1$ is also a successor and $\eta \in X$ for every set $X \in R(\xi, \alpha)$. Thus $\mathcal{R}(\xi, \alpha)$ is isomorphic to $\mathcal{S}^*(\eta, \beta) (\subseteq \mathcal{S}^*(\lambda, \beta))$ and so by the induction hypothesis there is a strictly increasing map $f_{\xi,\alpha}$ from $\mathcal{R}(\xi, \alpha)$ into $\mathcal{S}(\kappa, \kappa)$.

Case 2. α is a limit ordinal. In this case $R(\xi, \alpha) = \emptyset$ unless ξ is also a limit and has the same cofinality as α. Let $\xi_\rho (\rho < \mathrm{cf}(\alpha))$ be a strictly increasing sequence of ordinals cofinal in ξ. The map F defined by

$$F(X) = \langle X \cap \xi_\rho : \rho < \mathrm{cf}(\alpha) \rangle$$

is a strictly increasing map from $\mathcal{R}(\xi, \alpha)$ into the direct product $\mathcal{Q} = \bigotimes\{\mathcal{S}^*(\xi_\rho, < \alpha) : \rho < \mathrm{cf}(\alpha)\}$. Therefore, by the induction hypothesis, there is a strictly increasing map from $\mathcal{R}(\xi, \alpha)$ into $\mathcal{Q}_1 = \bigotimes\{\mathcal{S}(\kappa, \kappa) : \rho < \mathrm{cf}(\alpha)\}$, the direct product of $\mathrm{cf}(\alpha)$ copies of $\mathcal{S}(\kappa, \kappa)$. Since $\mathrm{cf}(\alpha) \leq \kappa$, there is a strictly increasing map from $\mathcal{Q}_1$ into $\mathcal{S}(\kappa, \kappa)$ and the result follows. ■

4. Perfect comparability graphs.

As already defined in §1, a graph G is good if it contains a clique of cardinality $\chi(G)$, the chromatic number of G, and it is perfect if every induced subgraph is good. The class $\mathcal{P}$ of all perfect graphs is an initial segment of the class of all graphs, $\mathcal{G}$, quasi-ordered by embeddability. Consequently, if $\mathcal{L}$ is a class of graphs which is coinitial in the class $\mathcal{G} \backslash \mathcal{P}$, then a graph G is perfect if and only if it does not embed any graph $L \in \mathcal{L}$. In other words, the members of $\mathcal{L}$ are "obstructions" to a graph being perfect. The problem is to describe such a list $\mathcal{L}$ of obstructions which is as simple as possible. If we restrict our attention simply to the finite graphs, then of course there is only one choice for $\mathcal{L}$, namely the minimal members of $\mathcal{G}\setminus\mathcal{P}$. But even in this case the problem is still not solved. It is well known (see Berge [1]) that the odd cycles C_{2n+1} $(n \geq 2)$ and their complements are minimal finite graphs in $\mathcal{G}\setminus\mathcal{P}$, but it is an outstanding problem of graph theory to decide if there are any others. In the case of infinite graphs there is an added difficulty since there is no reason to suppose that a non-perfect graph should embed a minimal one.

Viewed in this context, Theorem 1.7 asserts that, if we add the one infinite minimal non-perfect graph K_Ω to the (unknown!) set of finite ones, then we obtain a list of obstructions which prevent a graph of countable chromatic number from being perfect.

Proof of Theorem 1.7. Let G be a graph with chromatic number $\chi(G) \leq \omega$. We have to show that G is perfect if and only if every finite subgraph is perfect and G does not embed K_Ω. The necessity is obvious from the definition of a perfect graph and the fact that K_Ω is not even good. We have to prove the sufficiency.

Suppose that the graph G satisfies the conditions of the theorem and that G_1 is an induced subgraph. We need to show that G_1 is good. Suppose first that the clique number of G_1 is finite, say $\psi(G_1) = n$. Since, by assumption, every finite subgraph of G is perfect, it follows that every finite subgraph of G_1 has chromatic number at most n. Therefore, by the compactness theorem of De Bruijn and Erdős [3], it follows that $\chi(G_1) \leq n$, and so $\chi(G_1) = \psi(G_1)$. Therefore, we may assume that $\psi(G_1) = \omega$ since $\psi(G_1) \leq \chi(G_1) = \omega$. The theorem follows from the following lemma. ∎

Lemma 4.1. *If a graph G contains cliques of arbitrarily large finite size, then either G contains an infinite clique or it embeds K_Ω.*

Lemma 4.1 is a very special case of either one of the canonization lemmas of Shelah or Erdős, Hajnal and Rado (see [7], lemmas 27.2 or 27.8). In fact, this special case also follows by a direct application of Ramsey's theorem (for quadruples). Despite this, we shall prove it as the special case of Lemma 4.2 below when $\mathcal{F}$ is the class of all finite complete graphs K_n $(n = 1, 2, \ldots)$ (an argument similar to this can also be found in Pouzet [20]).

The *age* of a graph G is the class $\mathcal{A}(G)$ of all finite graphs which embed into G . If G is a graph and $\mathcal{F} = \{F_i : i \in I\}$ is a family of finite graphs, we say that G is $\mathcal{F}$-indivisible if, for any partition of the vertex set $V(G)$ of G into finitely many parts $A_1, A_2, \ldots, A_n$, there is some i such that $\mathcal{F} \subseteq \mathcal{A}(G|A_i)$. Also, we define the *direct sum* and the *complete sum* of the (vertex-disjoint) graphs $F_i = (V_i, E_i)$ to be respectively $\bigsqcup\{F_i : i \in I\} = (W, E_1)$ and $\sum\{F_i : i \in I\} = (W, E_2)$, where $W = \bigcup\{V_i : i \in I\}$, $E_1 = \bigcup\{E_i : i \in I\}$ and $E_2 = E_1 \cup \{\{x_i, y_j\} : x_i \in V_i, y_j \in V_j, i \neq j\}$.

We make the following observations:

(4.1) If G is F-indivisible and $V(G) = A_1 \cup \ldots \cup A_n$ is any partition into finitely many parts, then some $G|A_i$ is F-indivisible.

(4.2) If G is F indivisible, $\mathcal{F} \in F$ and $n < \omega$, then there is $\mathcal{H} = \mathcal{H}(\mathcal{F}, n) \in \mathcal{A}(G)$ such that, for any partition of $\mathcal{H}$ into n parts, some part embeds $\mathcal{F}$.

(4.3) If G is F-indivisible and $\mathcal{F} \in F$, then there are disjoint subsets A, B in $V(G)$ such that $G|A \cong \mathcal{F}$, $G|B$ is F-indivisible and EITHER every point of A is joined to every point of B OR no point of A is joined to a point of B.

(4.1) is immediate from the definition and (4.2) follows easily by compactness. We prove (4.3) as follows. By (4.2) there is $A_1 \subsetneq V(G)$ such that $G|A_1 \cong \mathcal{H}(\mathcal{F}, 2)$. Partition the remaining vertices of $V(G)\backslash A_1$ into finitely many classes so that x, y belong to the same class if and only if $\{z \in A_1 : \{x, z\} \text{ an edge}\} = \{z \in A_1 : \{y, z\} \text{ an edge}\}$. Since A_1 is finite, there is some class B such that $G|B$ is F-indivisible by (4.1). Now partition the points of A_1 into two classes according as they are joined or not joined to a point (and hence all points) of B. The result follows since, by the choice of A_1, one part contains a set A such that $G|A \cong \mathcal{F}$.

Lemma 4.2. *Let $\mathcal{F}$ be a family of finite graphs and let $F_n (n < \omega)$ be an increasing sequence of members of $\mathcal{F}$ (i.e. $F_m \leq F_n$ for $m < n$). Then any infinite $\mathcal{F}$-indivisible graph G embeds either the direct sum or the complete sum of the F_n.*

Proof. By (4.3) we can successively choose subsets $A_0, B_0, A_1, B_1, \ldots$ of $V(G)$, so that A_{n+1}, $B_{n+1} \subseteq B_n$, $G|A_n \cong F_n$, $G|B_n$ is $\mathcal{F}$-indivisible and $A_n \cap B_n = \emptyset$, and also so that, for each n, either (i) no point of A_n is joined to a point of B_n, or (ii) every point of A_n is joined to every point of B_n. Let I_1, I_2 be respectively the sets of integers n such that (i) or (ii) holds. Since the F_n are increasing, we see that G embeds either the direct sum or the complete sum of the F_n according as I_1 or I_2 is infinite. ■

By Theorem 1.7 it follows that if a graph G is not perfect but every induced finite subgraph is good, then G embeds the non-perfect comparability graph K_Ω. We can ask the following question.

Problem 8. *If every induced countable subgraph of a non-perfect graph G is perfect, does G embed a non-perfect comparability graph?*

We conclude this section with an additional observation about perfect comparability graphs. The notions of a graph being good or perfect can be weakened and strengthened in the following ways. Let us say that the graph G is *nearly good* if the chromatic number $\chi(G) = \sup\{|X| : X \text{ is a clique in } G\}$, and *very good* if there is a partition $\mathcal{A}$ of $V(G)$ into independent sets and a clique which has a non-empty intersection with each member of $\mathcal{A}$. Then the graph is *nearly* (or *very*) *perfect* if every subgraph is nearly (or very) good.

A finite comparability graph is very perfect, and a comparability graph with no infinite independent set is perfect by the Erdős-Dushnik-Miller theorem [5]. Also a countable comparability graph is nearly perfect. The graph K_Ω is an example of a countable comparabilty graph which is nearly perfect but not perfect. The following is an example of a countable comparability graph which is perfect but not very perfect.

Example: Let $\mathcal{P}$ be the poset on $\{(m,n) : n \leq m < \omega\}$ ordered so that $(m,n) \leq (m',n') \Leftrightarrow$ EITHER $m = m'$ & $n \leq n'$, OR $m < m'$ & $n+2 \leq n'$. The map $f : \mathcal{P} \to \omega$ given by $f(m,n) = n$ is strictly increasing, and the image of any antichain is finite. Therefore, by the theorem of Milner & Sauer (see Corollary 1.11), it follows that, if X is any subset whose image under f is infinite, then X contains an infinite chain. From this we see that the comparability graph of $\mathcal{P}$ is perfect. On the other hand, this comparability graph is not very good. To see this suppose, for a contradiction, that $\mathcal{A}$ is an antichain decomposition of $\mathcal{P}$ and that C is a chain having a non-empty intersection with each member of $\mathcal{A}$. Then there are successive elements (m_1,n_1), (m_2,n_2) of C such that $m_1 < m_2$ and $n_1 + 2 \leq n_2$. The set $D = \{(m,n) : (m,n) < (m_2,n_2)\}$ contains the elements (m_2,n) for $n \leq n_2 - 1$, and these belong to n_2 different members of $\mathcal{A}$, say $A_1, \ldots, A_{n_2}$. The chain $C_1 = C(\geq (m_2,n_2))$ cannot intersect any of these A_i, and so each must have a non-empty intersection with $C_2 = C(< (m_2,n_2))$. But this is impossible since $|C_2| \leq n_1 + 1 < n_2$.

The following related question arose in discussions with R. Aharoni.

Problem 9. *If G is a comparability graph which has no infinite independent set, must G be very good?*

Proof of Theorem 1.9. The implications $(ii) \Longrightarrow (iii) \Longrightarrow (i)$ of Theorem 1.9, and $(iv) \Longrightarrow (ii)$ in the case that $\mathcal{P}$ is well founded, are all obvious. The proof that $(i) \Longrightarrow (ii)$ follows by induction on the rank of $\mathcal{P}$ from the following slightly stronger result. The proof that $(i) \Longrightarrow (iv)$ for well founded $\mathcal{P}$ is similar and we omit it.

Theorem 5.1. *Let $\mathcal{P} = \langle P, \leq \rangle$ be a partially ordered set of rank $r(\mathcal{P}) = \nu$ and suppose that $\mathcal{P}'$ contains a chain of cardinality $r(\mathcal{P}')$ whenever $\mathcal{P}'$ is an induced suborder having smaller rank $r(\mathcal{P}') < r(\mathcal{P})$. Then either (o) $\mathcal{P}$ contains a chain of cardinality ν, or (oo) $\mathcal{P}$ embeds Ω, or (ooo) there is an embedding $f : \eta \to \mathcal{P}$ of the rational chain into $\mathcal{P}$ such that $r(\mathcal{P}|I) = \nu$ whenever $I = \mathcal{P}(> f(q)) \cap \mathcal{P}(< f(q')$ and $q < q'$ in η.*

In the proof of Theorem 5.1 we shall frequently use without reference the following simple fact.

Lemma 5.2. *If the partially ordered set $\mathcal{P}$ has rank $r(\mathcal{P}) = \nu \geq \omega$, and if D and U are disjoint initial and final segments of $\mathcal{P}$, then*

$$\max\{r(\mathcal{P}|D), r(\mathcal{P}|U), r(\mathcal{P}|(P \setminus (D \cup U))\} = \nu.$$

Proof. If $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$ are respectively rankable antichain decompositions of $\mathcal{P}|D$, $\mathcal{P}|U$ and $\mathcal{P}|P \setminus (D \cup U)$, then $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$ is a rankable antichain decomposition of $\mathcal{P}$. ■

Proof of Theorem 5.1. We can assume that ν is infinite since (o) holds trivially if ν is finite. For brevity we shall write $r(X)$ instead of $r(\mathcal{P}|X)$ for any subset $X \subset P$. For $X \subset P$ and $a \in P$ define $X(< a) = \mathcal{P}(< a)) \cap X$ etc. Also, define

$$\mathcal{D}(X) = \{x \in X : r(X(< x)) < \nu\}, \mathcal{U}(X) = \{x \in X : r(X(> x)) < \nu\},$$

$$\mathcal{D}_1(X) = \{x \in X : r(X(\not> x)) < \nu\}, \mathcal{U}_1(X)\{x \in X : r(X(\not< x)) < \nu\}.$$

The sets $\mathcal{D}(X)$, $\mathcal{D}_1(X)$ are initial segments of $\mathcal{P}$ and $\mathcal{U}(X)$, $\mathcal{U}_1(X)$ are final segments.

We shall consider separately several different cases.

Case 1. $r(\mathcal{D}(X)) < \nu$ and $r(\mathcal{U}(X)) < \nu$ whenever $X \subset P$ and $r(X) = \nu$.

We will show that, in this case, (ooo) holds. Let $q_n (n < \omega)$ be a $1-1$ enumeration of the rationals. Choose $f(q_0) \in P \backslash (\mathcal{D}(P) \cup \mathcal{U}(P))$. Then $r(P(< f(q_0)) = r(P(> f(q_0)) = \nu$. Generally, suppose that $n < \omega$ and that $f(q_i) \in P$ has been defined so that

$$q_i < q_j \Longrightarrow f(q_i) < f(q_j) \tag{5.1}$$

holds for $i, j < n$. Suppose also so that $r(I) = \nu$ whenever I is one of the $n+1$ open intervals $I_0, \ldots, I_n$ of $\mathcal{P}$ determined by successive elements of the chain $\{f(q_i) : i < n\}$; *i.e.* if $\{f(q_i) : i < n\} = \{x_1, \ldots, x_n\}$ and $x_1 < \ldots < x_n$, then $I_0 = \mathcal{P}(< x_1)$, $I_1 = \mathcal{P}(> x_1) \cap \mathcal{P}(< x_2)$ *etc.*. Now there is some $s \leq n$ such that 5.1 remains true for $i, j \leq n$ with any choice for $f(q_n)$ in I_s. Since $r(\mathcal{D}(I_s))$ and $r(\mathcal{U}(I_s))$ are both less than ν, it follows by Lemma 5.2 that we can choose $f(q_n) \in I_s \backslash (\mathcal{D}(I_s) \cup \mathcal{U}(I_s))$. From the definition of $\mathcal{D}(I_s)$ and $\mathcal{U}(I_s)$, it follows that $r(I') = r(I'') = \nu$, where $I' = I(< f(q_n))$ and $I'' = I(> f(q_n))$. This inductively defines the map $f : \eta \to \mathcal{P}$ with all the required properties.

Case 2. There is $X \subset P$ such that $r(\mathcal{D}_1(X)) = \nu$ or $r(\mathcal{U}_1(X)) = \nu$.

We will show that (o) holds in this case. By symmetry it will be enough to prove this for the case when $r(\mathcal{D}_1(X)) = \nu$. Since $\mathcal{D}_1(\mathcal{D}_1(X)) = \mathcal{D}_1(X)$, we may also assume that $X = \mathcal{D}_1(X)$.

Suppose first that EITHER (a) ν is regular OR (b) ν is singular and there is $\nu_1 < \nu$ such that $r(X(\not> x)) \leq \nu_1$ for all $x \in X$. Let $\alpha < \nu$ and suppose that we have already chosen $x_\beta \in X$ for $\beta < \alpha$ so that $x_0 < x_1 < \ldots < x_\beta < \ldots$ Since the sets $X(\not> x_\beta)(\beta < \alpha)$ are increasing initial segments of X, it follows, in either of the cases (a) or (b), that $r(\cup\{X(\not> x_\beta) : \beta < \alpha\}) \leq |\alpha| \sup\{r(X(\not> x_\beta)) : \beta < \alpha\} < \nu$. Hence there is $x_\alpha \in X \backslash \cup \{X(\not> x_\gamma) : \gamma < \alpha\}$, and it follows by induction that X contains a chain of order type ν.

Therefore, we may assume that ν is a singular cardinal and that, for any $\nu_1 < \nu$, there is $x \in X$ such that $r(X(\not> x)) > \nu_1$. Let $\nu_\alpha(\alpha < \mathrm{cf}(\nu))$ be an increasing sequence of cardinals cofinal in ν, and let $Y_\alpha = \{x \in X : r(X(\not> x)) \leq \nu_\alpha\}$. Since the sets $Y_\alpha(\alpha < \mathrm{cf}(\nu))$ are increasing initial segments of X whose union is X, it follows that $\nu = r(X) \leq \sum\{r(Y_\alpha) : \alpha < \mathrm{cf}(\nu)\}$. If, for some $\alpha < \mathrm{cf}(\nu)$, we have $r(Y_\alpha) = \nu$, then Y_α contains a chain of order type ν by the argument used for (b) above. Therefore, we may assume that $r(Y_\alpha) < \nu$ for all $\alpha < \mathrm{cf}(\nu)$, and also that, for any cardinal $\mu < \nu$, there is some $\alpha < \mathrm{cf}(\nu)$ such that $r(Y_\alpha) \geq \mu$. Let $\xi < \mathrm{cf}(\nu)$. Choose $\alpha < \mathrm{cf}(\nu)$ so that $r(Y_\alpha) > \nu_\xi^+$. By the hypothesis of the theorem there is a chain C in Y_α of cardinality $r(Y_\alpha)$ and we can choose $y \in C$ such that $|C(< y)| \geq \nu_\xi^+$. Note that $X(< y) \supset Y_\xi$, for if there is $z \in Y_\xi \backslash X(< y)$ then $X(\not> z) \supset X(< y) \supset C(< y)$, and so $r(X(\not> z)) \geq |C(< y)| > \nu_\xi$, and this contradicts the fact that $z \in Y_\xi$. Therefore,

$$(\forall \xi < \mathrm{cf}(\nu))(\exists y \in X)(Y_\xi \subset X(< y)). \tag{9.1}$$

Also, if $\xi < \mathrm{cf}(\nu)$ and $x \in Y_\xi$, then $r(X(\not> x)) \leq \nu_\xi$ and $r(X(> x)) = \nu$ and hence there is $\zeta < \mathrm{cf}(\nu)$ such that $r(X(> x) \cap Y_\zeta) \geq \nu_\xi$ and so, by hypothesis, $X(> x) \cap Y_\zeta$ contains a chain of cardinality ν_ξ. Thus, for all $\xi < \mathrm{cf}(\nu)$ and $x \in Y_\xi$, there exists $\zeta < \mathrm{cf}(\nu)$ such that

$$(X(> x) \cap Y_\zeta \text{ embeds a chain or cardinality } \nu_\xi). \tag{9.2}$$

We show that X contains a chain of cardinality ν by inductively choosing elements $x_\beta \in X$ for $\beta < \mathrm{cf}(\nu)$ so that (i) $x_\gamma < x_\beta$ for $\gamma < \beta$ and (ii) $X(> x_\beta) \cap X(< x_{\beta+1})$ contains a chain of cardinality ν_β. Let $\alpha < \mathrm{cf}(\nu)$ and suppose we have already defined x_β for $\beta < \alpha$ so that (i) holds for $\beta < \alpha$ and (ii) holds for $\beta + 1 < \alpha$. Suppose α is a limit. There is some $\xi < \mathrm{cf}(\nu)$ such that $x_\beta \in Y_\xi$ for all $\beta < \alpha$ and so by (9.1) there is $x_\alpha \in X$ such that $Y_\xi \subset X(< x_\alpha)$ and so (i) holds for $\beta \leq \alpha$ and (ii) holds for $\beta + 1 \leq \alpha$. Now suppose that $\alpha = \delta + 1$ is a successor. There is $\xi < \mathrm{cf}(\nu)$ such that $\nu_\delta < \nu_\xi$ and $x_\delta \in Y_\xi$. By (9.2) there is $\zeta < \mathrm{cf}(\nu)$ such that $X(> x_\delta) \cap Y_\zeta$ contains a chain of size $\nu_\xi \geq \nu_\delta$. By (9.1) there is $x_\alpha \in X$ such that $Y_\zeta \subset X(< x_\alpha)$. Then (i) holds for $\beta \leq \alpha$ and (ii) holds for $\beta + 1 \leq \alpha$.

Case 3. For all $X \subset P$ such that $r(X) = \nu$ there are $x, y \in X$ such that

$$r(X(> x) \backslash X(> y)) = r(X(> y) \backslash X(> x)) = \nu.$$

In this case there is an embedding of the binary tree $\langle 2^{<\omega}, \subset \rangle$ into $\mathcal{P}$ and so $\mathcal{P}$ embeds Ω. Indeed, there is $x_\emptyset \in P$ such that $\mathcal{P}(> x_\emptyset) = \nu$. Let $n < \omega$ and suppose that $x_f \in P$ has been defined for all $f \in 2^{\leq n}$ so that $r(P_f) = \nu$, where

$$P_f = \mathcal{P}(> x_f) \backslash \cup \{\mathcal{P}(> x_g) : g \in 2^{\leq n} \text{ and } g \not\subseteq f\}.$$

By assumption, for each $f \in 2^v$, there are x_{f*0} and x_{f*1} in P_f such that $P(f, i) = P_f(> x_{f*i}) \backslash P_f(> x_{f*j})$ has rank ν for $\{i, j\} = \{0, 1\}$; the induction continues since $P_{f*i} = P(f, i)$.

Case 4. There is a set $X \subset P$ of rank ν such that either $X(> x) \backslash X(> y)$ or $X(> y) \backslash X(> x)$ has rank less than ν whenever $\{x, y\} \subset X$.

Let L denote the set of all pairs $(x,y) \in X \times X$ such that $r(X(> x)\backslash X(> y)) < \nu$. Since $X(> x)\backslash X(> z) \subseteq (X(> x)\backslash X(> y)) \cup (X(> y)\backslash X(> z))$, it follows that L is transitive and so there is a linear order $<<$ on X which extends L.

By Case 1, we may assume that either $r(\mathcal{D}(X)) = \nu$ or $r(\mathcal{U}(X)) = \nu$. By symmetry we may suppose that $r(\mathcal{D}(X)) = \nu$, and since $\mathcal{D}(\mathcal{D}(X)) = \mathcal{D}(X)$, we may suppose that $X = \mathcal{D}(X)$. By Case 2, we may also assume that $r(\mathcal{D}_1(X)) < \nu$, so that $\mathcal{D}_1(X\backslash\mathcal{D}_1(X)) = \emptyset$. Replacing X by $X\backslash\mathcal{D}_1(X)$, we may suppose further that $\mathcal{D}_1(X) = \emptyset$. We will show that, under these assumptions, $\mathcal{P}|X$ embeds Ω.

Let $n < \omega$ and suppose that we have already chosen n chains $C_0, \ldots, C_{n-1}$ such that $|C_i| = i+1$, say $C_i = \{x_{i0}, \ldots, x_{ii}\}$ with $x_{i0} < \ldots < x_{ii}$, and such that there is no order relation (in $\mathcal{P}$) between the elements of C_i and C_j if $i \neq j$. There is some index $t < n$ such that $x_{i0} << x_{t0}$ for all $i \neq t$ and $i < n$, so that $r(X(> x_{i0})\backslash X(> x_{t0}) < \nu$. Since $\mathcal{D}_1(X) = \emptyset$, it follows that $r(S) = r(X(\not> x_{t0})) \doteq \nu$, where

$$S = X(\not> x_{t0})\backslash \cup \{X(> x_{io}) : i < n,\, i \neq t\}.$$

Also, since $X = \mathcal{D}(X)$, the initial segment $T = \cup\{X(< x_{ii} : i < n\}$ of X has rank $r(T) < \nu$. therefore, $r(S\backslash T) = \nu$ and so there is a chain C_n of size $n+1$ in $S\backslash T$ and, by the definitions of S and T, there is no order relation between the elements of C_n and the elements of $C_0 \cup \ldots \cup C_{n-1}$. It follows by induction that $\mathcal{P}$ embeds Ω.

References

[1] C. Berge, *Graphs and hypergraphs*, North Holland, Amsterdam, 1973.

[2] R. Bonnet and M. Pouzet, Extensions et stratifications d'ensembles disperses, *CRAS*, **268**-Serie A-(1969), 1512–1515.

[3] N. G. de Bruijn and P. Erdős, On a combinatorial problem, *Nederl. Akad. Wetensch. Proc.* **52**(1948), 14–17 = *Indagationes Math.* **11**(1948), 46–49.

[4] R. P. Dilworth, A decomposition theorem for partially ordered sets, *Annals of Math.* **31**(1950), 161–166.

[5] B. Dushnik and E. W. Miller, Partially ordered sets, *Amer. J. of Math.* **63**(1941), 600–610.

[6] P. Erdős, F. Galvin and A. Hajnal, On set systems having large chromatic number and not containing prescribed subsystems, in: *Infinite and Finite Sets* (eds.: A. Hajnal et. al), Colloq. Math. Soc. János Bolyai, **11**, North-Holland, Amsterdam, 1975, 425–513.

[7] P. Erdős, A. Hajnal, A. Máté and R. Rado, *Combinatorial Set Theory: Partition relations for cardinals,* Akadémiai Kiadó. Budapest 1984.

[8] G. Fodor, Proof of a conjecture of P. Erdős, *Acta Sci. Math.* **14**(1952), 219–227.

[9] R. Fraïssé, *Theory of Relations*, Studies in logic and the foundations of mathematics; **118**, North Holland, Amsterdam, 1986.

[10] F. Galvin, Private communication, March 1988.

[11] F. Hausdorff, Grundzüge einer Theorie der geordneten Mengen, *Math. Ann.* **65**(1908), 435–505.

[12] T. Hiraguchi, On the dimension of partially ordered sets, *Sci. Rep. Kanazawa Univ.* **1**. MR 17, p 1045.

[13] M. Kojman and S. Shelah, *Existence of universal models,* (1988)

[14] G. Kurepa, *Ensembles ordonnés et ramifiés,* Thése, Paris 1935. [Published in *Math. Belgrade* **4**(1935), 1–138.]

[15] E. C. Milner and M. Pouzet, A note on the dimension of a poset, *Order* **7**(1990), 101–102.

[16] E. C. Milner and N. Sauer, On chains and antichains in well founded partially ordered sets, *J. London Math. Soc.* (2) **24**(1981), 15–33.

[17] W. Mitchell, Aronszajn trees and the independence of the transfer property, *Annals of Math. Logic* **5**(1972), 21–46.

[18] M. A. Perles, On Dilworth's theorem in the infinite case, *Israel J. Math.* **1**(1963), 108–109.

[19] M. Pouzet, Chaines d'un ensemble partiellement bien ordonné, *Pub. Dept. Math. Lyon* **16**(1)(1979), 21–26.

[20] M. Pouzet, Relations impartible, *Dissertationes Math.* **103**(1981), 1–48.

[21] M. Scheepers, Private communication, September, 1989.

[22] D. Schmidt, The relation between the height of a well founded partially ordering and the order types of its chains and antichains, *J. Combinatorial Theory (B)* **31**(1981), 183–189.

[23] S. Shelah, Decomposing uncountable squares to countably many chains, *J. Combinatorial Theory (A)* **21**(1976), 110–114.

[24] S. Shelah, Independence results, *J. Symbolic Logic* **45**(1980), 563–573.

[25] Z. Szentmiklóssy, Private communication, January, 1991.

[26] S. Todorčević, Partitioning pairs of countable ordinals, *Acta Mathematica,* **159** (1987), 261–294.

E. G. Milner

University of Calgary,
Alberta,
Canada

M. Pouzet

Université Claude Bernard,
Lyon I,
France