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Connected Triangle-free (1,2)-realizable Graphs

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ABSTRACT

A finite graph H is (1,2)-realizable if there exists a graph such that all its vertices have i-neighbourhood (subgraph induced on vertices at the distance i), $i = 1, 2$, isomorphic to H. In this paper all connected triangle-free (1,2)-realizable graphs are characterized.

0. Introduction

Vertices at the distance i from a vertex of a graph induce i-neighbourhood of that vertex. Zykov [14] posed a problem now generally known as Trahtenbrot-Zykov problem: Given a finite graph H, does there exist a graph with only vertices having 1-neighbourhood isomorphic to H? If such a graph exists, it is called 1-realization of H, and H is called 1-realizable. The problem attracted many authors, among others Blass, Harary and Miller [2], Brown and Connelly [3,4], Hall [9], Hell [11]. Bulitko [8] showed (cf. also Bugata [5]) that there exists no algorithm to recognize 1-realizable graphs. No analogous result concerning 1-realizability by finite graphs is known till now.

A natural generalization of the mentioned problem consists in analyzing i-realizability for a positive integer i (see Bielak [1]). According to Bugata, Horňák and Jendroľ[6] 1-realizability of a non-empty graph [H] is equivalent to 2-realizability of the graph $\bar{H}+K_1$ (formed by adding a new vertex to the complement of H and joining it to all vertices of H); thus the problem of 2-realizability is algorithmically not solvable as well. However, by Bielak [1] any finite graph H is i-realizable for $i \geq 3$ (e.g. by the composition $C_{2i}[H]$ of the graphs C_{2i} and H).

The problem of (1,2)-realizability was introduced by Bugata, Horňák and Nagy [7]. They showed that any (1,2)-realizable graph non-isomorphic to $2K_1$ has a unique connected (1,2)-realization and yielded the construction of this (1,2)-realization. They found a necessary and sufficient condition of (1,2)-realizability (which immediately proves the algorithmic solvability of the problem) and applied it in the analysis of simplest connected regular (1,2)-realizable graphs.

In the present paper we characterize the set of all connected triangle-free (1,2)-realizable graphs. First we show that C_5 is the only non-bipartite connected triangle-free (1,2)-realizable graph. In Section 3. an infinite class of graphs is determined to be the set of all bipartite triangle-free (1,2)-realizable graphs.

1. Basic definitions and notation

A graph will mean an undirected graph without loops and multiple edges. For a graph G let $V(G)$ be its vertex set with cardinality $v(G)$, $E(G)$ its edge set (a subset of the set of all 2-element subsets of $V(G)$) and let x, y, v, w be vertices of G. The notation for basic notions (see e.g. Harary [10]) of the graph theory will be as follows: $\deg_G(x)$ — the degree of x in G, $\Delta(G)$ — the maximal degree of a vertex of G, $d_G(x,y)$ — the distance between x and y in G, $e_G(x)$ — the eccentricity of x in G, $r(G)$ — the radius of G, $G\langle U\rangle$ - the graph induced in G by a set $U \subseteq V(G)$. For some important graphs on m vertices we shall use the symbols K_m — the complete graph, C_m — the cycle, mK_1 — the graph without edges.

For a non-negative integer i define $V_i(x,G) = \{z \in V(G); d_G(x,z) = i\}$. The graph $N_i(x,G) = G\langle V_i(x,G)\rangle$ is said to be i-neighbourhood of x in G. $G_1 \simeq G_2$ will describe the fact that G_1 is isomorphic to G_2. If $\deg_G(x) = 0$, x is called an isolated vertex (in G). Edges $\{x,y\}$ and $\{v,w\}$ are said to be independent, iff $\{x, y\} \cap \{v, w\} = \emptyset$ and $\{\{x, v\}, \{x, w\}, \{y, v\}, \{y, w\}\} \cap E(G) = \emptyset$.

A finite graph H is said to be (1,2)-*realizable* if there exists a graph G such that for every $x \in V(G)$ the graphs $N_1(x, G)$ as well as $N_2(x, G)$ are isomorphic to H; if so the graph G is an (1,2)-*realization* of H.

Analysing (1,2)-realizability of a graph H the following objects are important: the graph $C(x, H)$ defined for $x \in V(H)$ by

$$V(C(x,H)) = V(H),$$

$$E(C(x,H)) = E(H) \cup \bigcup_{i=2}^{\infty} \{\{y,z\} : y \in V_1(x,H), z \in V_i(x,H)\} - \{\{y,z\} : y \in V_1(x,H), z \in V_2(x,H)\}$$

and the equivalence relation

$$R_{1^-}(H) = \{(y,z) : y, z \in V(H), d_H(y,z) \le 1, N_1(y,H) - \{z\} = N_1(z,H) - \{y\}\}.$$

2. Connected triangle-free non-bipartite (1,2)-realizable graphs

Theorem 2.1. *A finite graph H is (1,2)-realizable if and only if $v(H) \neq 1$, $R_{1^-}(H) = \{(u,u) : u \in V(H)\}$ and $C(u,H) \simeq H$ for each $u \in V(H)$.*

The following assertions are immediate corollaries of Theorem 2.1:

Corollary 2.2. *If H is a non-empty (1,2)-realizable graph, then $\Delta(H) \le v(H) - 2$.*

Corollary 2.3. *If H is a non-empty connected (1,2)-realizable graph, $u \in V(H)$ and $\deg_H(u) = \Delta(H)$, then $e_H(u) = r(H) = 2$.*

Proofs of 2.1, 2.2 and 2.3 can be found in [7]. ■

Corollary 2.4. *If G is a connected (1,2)-realizable graph, $v(G) \ge 2$, then for any pair of its distinct non-adjacent vertices x, y $V_1(x,G) \neq V_1(y,G)$.*

Proof. Assume that there exist $x, y \in V(G)$, $x \neq y$, $\{x, y\} \notin E(G)$, $V_1(x,G) = V_1(y,G)$. Consider $C(x,G)$: it is isomorphic to G by Theorem 2.1, and y is its isolated vertex — a contradiction with connectedness of G. ■

Corollary 2.5. *If G is an (1,2)-realizable triangle-free graph, then G has no pair of independent edges.*

Proof. . If $\{x_1, x_2\}$, $\{y_1, y_2\}$ are independent edges of G, then $G\langle x_1, x_2, y_1\rangle$ is a triangle in $C(y_2, G)$. According to Theorem 2.1 $C(y_2, G) \simeq G$, i.e. G contains a triangle — a contradiction with the assumptions. ■

Theorem 2.6. *C_5 is the only connected non-bipartite triangle-free (1,2)-realizable graph.*

Proof. Let H be a connected non-bipartite triangle-free graph and let $v \in V(H)$, $\deg_H(v) = \Delta(H)$. Clearly, $\deg_H(v) \geq 2$ and by Corollary 2.3 $e_H(v) = 2$. Let us denote $H_1 = N_1(v, H)$ and $H_2 = N_2(v, H)$. Note that H_1 is a graph without edges. Clearly, any two edges from different components of H_2 are independent in H, hence H_2 has at most one non-trivial component. Such component exists, since H contains an odd circle.

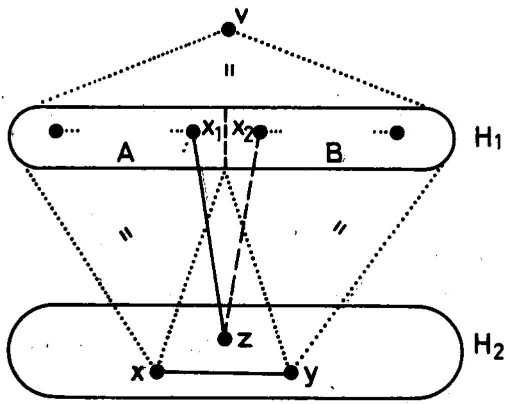

Figure 1.

Let $\{x, y\}$ be an edge of H_2. Then any vertex w of H_1 is connected by an edge with at least one of the vertices x, y (otherwise the edges $\{x, y\}$ and $\{v, w\}$ would be independent), and none of the vertices of H_2 is connected with both of them (H is a triangle-free graph). So $\{A, B\}$, where $A = V(H_1) \cap V_1(x, H)$, $B = V(H_1) \cap V_1(y, H)$, is a decomposition of $V(H_1)$ (note that A and B are non-empty).

Suppose that there exists a vertex $z \in V(H_2)$, with $\deg_{H_2}(z) = 0$ (see Fig.1). Then there is a vertex $x_1 \in V(H_1)$ adjacent to z (it follows from connectedness of H), and a vertex $x_2 \in V(H_1)$ non-adjacent to z

(non-existence of such a vertex would mean $V_1(v, H) = V_1(z, H)$ — a contradiction with Corollary 2.4). Suppose (without loss of generality) that $x_1 \in A$, $x_2 \in B$. Then the edges $\{y, x_2\}$ and $\{z, x_1\}$ are independent — a contradiction with Corollary 2.5. Hence, H_2 is a non-trivial connected graph.

It is easy to see that for any vertex w of H_2 exactly one of two possibilities holds:

(i) $N_1(w, H_1) = A$,

(ii) $N_1(w, H_1) = B$.

Let us denote

$$N(A) = \{w \in V(H_2) : V_1(w, H_1) = A\},$$
$$N(B) = \{w \in V(H_2) : V_1(w, H_1) = B\}$$

(note that $x \in N(A), y \in N(B)$).

Any two distinct vertices in A (in B) would have equal 1-neighbourhoods, which contradicts to Corollary 2.4. So $\text{card}(A) = \text{card}(B) = 1$. Then $\Delta(H) = 2$, it means $\text{card}(N(A)) = \text{card}(N(B)) = 1$. Therefore, H must have no more than 5 vertices. The only triangle-free non-bipartite graph, which has at most 5 vertices is C_5.

The (1,2)-realization of C_5 of Fig. 2. completes the proof. ■

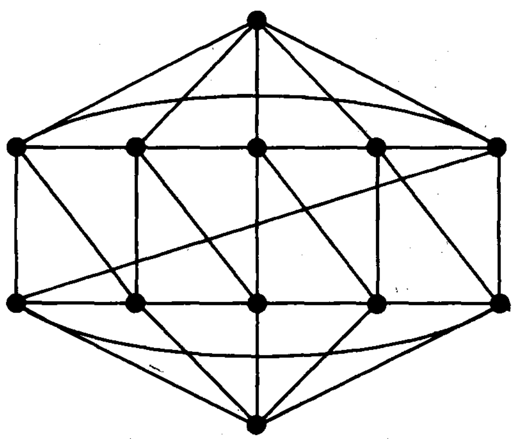

Figure 2.

3. Connected bipartite (1,2)-realizable graphs

For an integer $k \geq 2$ define the graph F_k as follows:

$$V(F_k) = \{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_k\}$$
$$E(F_k) = \{\{a_i, b_j\} : i + j \leq k + 1\}.$$

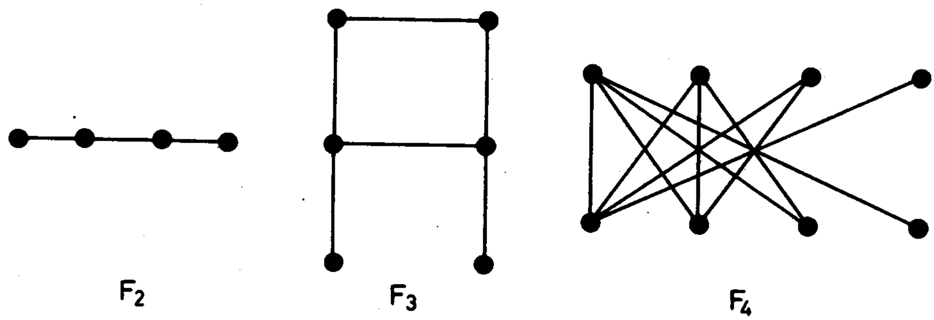

Figure 3.

Clearly, for every $k \geq 2$, F_k is a connected bipartite graph, and the mapping $\mathbf{P}_k : V(F_k) \to V(F_k)$, which switches a_i and b_i for $i = 1, \ldots, k$, is an automorphism of F_k.

Denote $F = \{F_k; k \geq 2\}$. The smallest graphs belonging to F are shown in Figure 3.

Theorem 3.1. *F is the set of all (up to isomorphism) non-empty connected bipartite (1,2)-realizable graphs.*

Proof. Let k be an integer, $k \geq 2$. Trivially, $R_{1^-}(F_k)$ is the identity relation for every $k \geq 2$. It is easy to check that the mappings $\mathbf{Q}_j : V(F_k) \to V(F_k)$ for $j = 1, .., k$ are isomorphisms between F_k and $C(a_j, F_k)$:

a)

$$\mathbf{Q}_1(a_1) = a_1$$
$$\mathbf{Q}_1(a_m) = a_{k-m+2} \quad \text{for } m = 2, \ldots, k$$
$$\mathbf{Q}_1(b_m) = b_{k-m+1} \quad \text{for } m = 1, \ldots, k$$

b) for $j \in \{2, \ldots, k-1\}$:

$$\mathbf{Q}_j(a_m) = a_{k-m+j+1} \quad \text{for } m = j+1, \ldots, k$$
$$\mathbf{Q}_j(b_m) = a_{-k+m+j-1} \quad \text{for } m = k-j+2, \ldots, k$$
$$\mathbf{Q}_j(a_m) = b_{k+m-j+1} \quad \text{for } m = 1, \ldots, j-1$$
$$\mathbf{Q}_j(b_m) = b_{k-m-j+2} \quad \text{for } m = 1, \ldots, k-j+1$$

c)

$$\begin{aligned} \mathbf{Q}_k(a_k) &= a_k, \quad \mathbf{Q}_k(b_1) = b_1 \\ \mathbf{Q}_k(a_m) &= b_{m+1} \quad \text{for } m = 1, \dots, k-1 \\ \mathbf{Q}_k(b_m) &= a_{m-1} \quad \text{for } m = 2, \dots, k \end{aligned}$$

The composition of isomorphisms $\mathbf{P}_k\mathbf{Q}_j\mathbf{P}_k$ is an isomorphism between F_k and $C(b_j, F_k)$.

It remains to show that an arbitrary connected bipartite (1,2)-realizable graph belongs to F. Let H be such a graph and $\{A, B\}$ a disjoint partition of its vertex set. Let us denote card$(A) = \alpha$, card$(B) = \beta$. Notice that $\alpha \geq 2$, $\beta \geq 2$.

From Corollary 2.5 follows that for any pair of vertices $x, y \in A$ ($x, y \in B$) at least one of the inclusions $V_1(x, H) \subseteq V_1(y, H), V_1(x, H) \supseteq V_1(y, H)$ holds: existence of vertices $x_1 \in V_1(x, H) - V_1(y, H)$ and $y_1 \in V_1(y, H) - V_1(x, H)$ would mean that $\{x, x_1\}$ and $\{y, y_1\}$ are independent edges.

All degrees of vertices in A (in B) are distinct: if $x_1, x_2 \in A (x_1, x_2 \in B)$, $\deg_H(x_1) = \deg_H(x_2)$, then $V_1(x_1, H) = V_1(x_2, H)$ — a contradiction with Corollary 2.4. Furthermore, $\alpha < \beta$ ($\beta < \alpha$) would yield repeated degrees of vertices in A (in B). Hence, $\alpha = \beta$.

Let a be a vertex with maximal degree in H. Suppose (without loss of generality) $a \in A$. According to Corollary 2.3 $e_H(a) = 2$. Therefore, $N_1(a, H) = B$, i.e. $\deg_H(a) = \beta = \alpha$. Let the vertices $a_1, \dots, a_\alpha$ of A and $b_1, \dots, b_\alpha$ of B fulfill the following condition: $i < j$ implies $\deg_H(a_i) < \deg_H(a_j)$ and $\deg_H(b_i) < \deg_H(b_j)$, $i, j = 1, \dots, \alpha$. Then $\deg_H(a_i) = \deg_H(b_i) = i$, $V_1(a_i, H) = \{b_1, .., b_{\alpha-i+1}\}$ and $V_1(b_i, H) = \{a_1, .., a_{\alpha-i+1}\}$ for $i = 1, .., \alpha$. Hence, $H \simeq F_\alpha$. ■

Remark 3.2. *The previous assertions together with those of [12] determine the set of all triangle-free (1,2)-realizable graphs as*

$$\{iK_1 : i \geq 2\} \cup \{G[iK_1] \cup (i-1)K_1 : i \geq 1, G \in F \cup \{C_5\}\} \cup K_0.$$

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