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Counting Domino Arrangements

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ABSTRACT

Consider the Cartesian product of the complete graphs K_k and K_n. The number of the Hamiltonian cycles of such graphs is investigated. A closed form is given for the case $k = 2$ and upper and lower estimates for general k and n.

1. Introduction

This paper is basically concerned with the following problem and its generalizations:

We are given kn dominos of the ordinary type, but the two fields of all dominos are coloured by black and white. The black fields are numbered from 1 to k, and the white ones are numbered from 1 to n, all combinations appearing exactly once. These dominos are to be arranged in a circle such that all neighbours should share either the black or the white number. (Such arrangements obviously exist.)

The main question is: *How many such arrangements are there?*

The special case of $k = 3$, $n = 4$ was asked by Rosemary Fraser of the Shell Centre of Education (Nottingham), in connection with school practice using Dienes set of the same parameter. A search for literature showed that the same problem has been attacked successfully by D. Ball [1]. Here we report about our work on a generalization of this special case. This renewed attempt is justified by the more general setting of the problem.

As a first step, let us note that our problem can be formulated on the language of graphs: We are given a five-regular (in general: $k+n-2$) graph of 12 (in general: kn) points with given incidence matrix. Now the number of Hamiltonian cycles is asked. This formulation leads us to the general case.

Let $G(k,n)$ be the Cartesian product of the complete graphs K_k and K_n. The question is that how many Hamiltonian cycles $G(k,n)$ has.

In this paper we give a closed form for

$$f(k,n) = \text{number of oriented Hamiltonian cycles in } G(k,n)$$

in the case of $k = 2$, and upper and lower estimates in the general case.

The original problem, as arisen from the school practice, has small size. This would allow an answer by checking all permutations systematically on a computer, using an algorithm of Knuth [2]. Such a program could be prepared even by school children to solve the case $k = 3$, $n = 4$ even on a Commodore type home computer. A BASIC program yields within acceptable time the number of the Hamiltonian cycles of the graph $G(3,4)$. The resulting $f(3,4) = 3132$ value is the same as was obtained by Derek Ball, [1].

Such a direct method is unfeasible on home computers, since it unnecessarily checks all permutations and not only the potential Hamiltonians. The searching time can considerably decreased by moving up and down on a searching tree, but even such an acceleration cannot help to solve case of larger kn.

2. The case $k = 2$ (two complete graphs)

This is a relatively simple case.. We have two complete graphs and any Hamiltonian cycle connects them by an even number of edges, which we call bridges. Suppose there are exactly $2r$ such edges, where obviously $r \geq 1$, $2r \leq n$. Let us denote the labels which are connected by these bridges by $L_1, L_2, \ldots, L_{2r}$. Now consider the complete graph with the $2r$

vertices $L_i : i = 1, 2, \ldots, 2r$. This contains $(2r-1)!$ Hamiltonian cycles and it is easy to see, that all of them corresponds exactly to two possible cyclical chainings of the bridges. Since there are $\binom{n}{2r}$ possible ways to choose the $2r$ labels, therefore there are $2\binom{n}{2r}(2r-1)!$ different cyclical ways of choosing the $2r$ ordered bridges.

Let us fix an ordered set of $2r$ bridges arbitrarily. Then in both copies of the complete n-graphs we have still $n - 2r$ unvisited vertices. These can be toured independently in the two graphs, thus it suffices to consider what happens in one of them. Since all of them are to be visited just once, therefore they can be classified according to which entry bridge they belong to.

This means that the $n - 2r$ points fall into r classes, where some classes may contain no vertices. Besides, this classes are ordered according to the numbering of the bridges. As is well known, the number of such classifications is given by

$$\sum \frac{(n-2r)!}{k_1! \ldots k_r!} k_1! \ldots k_r!,$$

where the summation goes over the set of compositions

$$K(n,r) = \{k_1 + \ldots + k_r = n - 2r; \quad k_i \geq 0\}.$$

This yields the ugly form

$$\sum_r \left[\sum_{K(n,r)} (n-2r)! \right]^2 2\binom{n}{2r}(2r-1)!.$$

From here one gets, after straightforward manipulation, the following assertion:

Theorem 1.

$$f(2,n) = \sum_{r=1}^{n/2} \frac{(n-r-1)!^2}{r!} \cdot \frac{n!}{(r-1)!(n-2r)!} \qquad \blacksquare$$

The formula gives the numbers 2, 6 and 60 for the cases $n = 2,\ 3,\ 4$, which can be checked directly.

3. Asymptotics

Since the number of the "good arrangements" cannot be larger than the number of all cyclic permutations, we have a rather trivial upper bound:

$$f(k,n) \leq (kn-1)!$$

Surprisingly, this bound gives the exact rate of growing to infinity of $\log f(k,n)$ for any fixed k. This follows from a trivial lower bound:

$$f(k,n) \geq (k-1)![(n-1)!]^{k-1}(n-2)!$$

In order to show this we characterize the vertices of our graph by an ordered pair of integers, where the first one gives the order of the corresponding complete graph (it runs from 1 to k) and the second gives the label of the vertex within this graph (therefore it runs from 1 to n). With this notation a large set of Hamiltonians can be specified by sequencing all pairs in blocks with identical first number as follows.

Let the first pair be $(1,1)$ and place all the n pairs with first number 1 in the first block in any order.

Having completed the i-th block, $i \leq n-2$, choose any integer what was not yet used as first number and use it as a common first number of the next block. The second number of the leading pair in this block be identical with the second number of the last pair in the last block, the other $n-1$ second numbers in this block can be chosen in any order. This can be applied to the k-th (last) block, with the difference that the second number of the very last pair must be 1.

It is easy to see, that all sequences obtained this way represent a Hamiltonian. One gets the desired bound by simply counting them.

Combining the two inequalities we have by standard technique the following

Lemma 1. *For fixed k it holds true that*

$$\lim_n \frac{\log f(k,n)}{kn\log n} = 1. \qquad \blacksquare$$

For $k = 2$ Theorem 1 allows us to establish a more exact order of magnitude. Indeed, it immediately follows from there that

$$f(2,n) < n!(n-2)!\sum_{r=1}^{\infty}\frac{1}{r!}$$

yielding the upper bound

$$f(2,n) < (e-1)n!(n-2)!\,.$$

Here the constant $e-1$ can not be improved, as the following argument shows. For any $\varepsilon > 0$ there is a threshold M such that

$$\sum_{r=1}^{m}\frac{1}{r!} > (e-1)(1-\varepsilon),$$

provided $m > M$. For $n > 2M$ Theorem 1 implies

$$f(2,n) > n!(n-2)!\sum_{r=1}^{M}\frac{1}{r!}\frac{(n-M-1)!}{(n-2)!}.$$

Now, for any $\delta > 0$ there is a threshold n_0 such that for $n > n_0$ we have

$$\frac{(n-M-1)!}{(n-2)!} > 1-\delta$$

and therefore for large enough n the inequality

$$f(2,n) > (e-1)(1-\varepsilon)(1-\delta)n!(n-2)!$$

holds true. Summing up, we have proved

Theorem 2.

$$\lim_{n}\frac{f(2,n)}{n!(n-2)!} = e-1. \qquad \blacksquare$$

References

[1] D. Ball, Using graphs to count logic block chains, *The Mathematical Gazette* **61**(1977), 288–291.

[2] D. Knuth, *The art of computer programming,* Vol. 1., Addison-Wesley, 1981.

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