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Polyhedral and Eigenvalue Approximations of the Max-Cut Problem

S. POLJAK*

ABSTRACT

We survey lower and upper bounds on the max-cut problem and the maximum bipartite subgraph problem, and discuss some connections between them. As a new result we prove that two distinct relaxations of the cut polytope lead to the same polyhedral upper bound on the maximum cut of a nonnegatively weighted graph. We observe that the ratio $\sigma(G)/mc(G)$ between the polyhedral upper bound $\sigma(G)$ and the actual size $mc(G)$ of the max-cut can be arbitrarily close to 2. This shows that the polyhedral upper bound can be, in a certain sense, as bad as possible.

Next we consider an eigenvalue upper bound $\varphi(G)$ on the max-cut. We show that $\varphi(G) \leq 1.131\sigma(G)$ for every G, i.e. the eigenvalue bound is never worse than a small multiple of the polyhedral bound. The converse is not true, since $\sigma(G)/\varphi(G)$ may get arbitrarily close to 2 as shown on the example of Ramanujan graphs. These graphs have the following two properties: (i) large girth, and (ii) they contain only small bipartite subgraphs. The latter property follows by an application of the eigenvalue upper bound. The existence of such graphs was earlier proved by Erdős, but no explicit construction was known so far.

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1. The max-cut problem and lower bounds

Let $G = (V, E)$ be a graph together with a nonnegative edge-weight function $w = (w_e)$. We assume that $V = \{1, \ldots, n\}$, and $w_{ij} = 0$ for $ij \notin E$. For a set $S \subset V$, the edge-set $\delta S = \{ij \mid i \in S, j \notin S\}$ is called *a cut* induced by S, and we set $w(S) = \sum_{ij \in \delta S} w_{ij}$. *The maximum cut problem* is to maximize $w(\delta S)$ for a given weighted graph G. We set $mc(G, w) = \max_{S \subset V} w(\delta S)$, and call the value the *max-cut* of G.

If $G = (V, E)$ is a simple (nonweighted) graph, we identify it with a weighted graph with all edge weights equal one. The cardinality version of the max-cut problem is known as *the maximum bipartite subgraph* problem. We write sometimes $mc(G)$ instead of $mc(G, w)$ when there is no danger of confusion. In particular, if G is an unweighted graph, then $mc(G)$ denotes the size of the maximum bipartite subgraph. The maximum bipartite subgraph problem, and hence also the max-cut problem, are NP-complete (see [14]).

A well known simple lower bound is the following one.

Lemma 1. *We have $mc(G, w) \geq \frac{1}{2} \sum_{e \in E} w_e$ for every weighted graph G.*

Proof. Assume that $w(\delta S)$ is maximum for $S \subset V$, and consider the bipartition $(S, V \backslash S)$. Then, for every vertex, the sum of the edge-weights on the edges leading to the opposite class is greater or equal than that the sum on the edges leading to the same class, since otherwise switching the vertex to the opposite class would increase the value of the cut. ∎

The lower bound of Lemma 1 is quite sufficient for the purpose of this paper. In fact, it is not possible to increase it substantially because the following result of Erdős holds true.

Theorem 2. *([10]; see also [11] and [12]). For every ℓ, there exists an infinite family of graphs G with girth greater than ℓ, and $mc(G) \leq \frac{m}{2} + m^{1-\varepsilon_\ell}$ where ε_ℓ, $0 < \varepsilon_\ell < 1$, is a constant and m is the number of edges of G.* ∎

Theorem 2 was proved by a probabilistic method, and no explicit class of graphs with this property was yet known. We show in Section 3 that the Ramanujan graphs present such a class.

Let us recall some known bounds on the size of maximum bipartite subgraphs. Edwards [9] proved that $mc(G) \geq \frac{1}{2}m + \frac{1}{4}(n-1)$ for a connected graph with n vertices and m edges. A polynomial time algorithm that guarantees this bound was given in [22], and generalized to the weighted case in [23].

A well known symmetrization argument yields a lower bound $mc(G,w) \geq (\frac{1}{2}+\frac{1}{2n})\sum_{e\in E} w_e$. The symmetrization can be algorithmized by a method of [16], using a doubly transitive group (see [24] for an application to the max-cut.)

Erdős and Lovász (see [11]) proved by probabilistic methods that every triangle-free graph with m edges contains a bipartite subgraph with at least

$$\frac{1}{2}m + cm^{2/3}\left(\frac{\log m}{\log\log m}\right)^{1/3}$$

edges, for some positive constant c. In [24], we improved their bound to

$$\frac{1}{2}m + c(m\log m)^{2/3},$$

and showed that a bipartite subgraph with at least that many edges can be constructed by a polynomial time algorithm. Some other lower bounds on the size of the maximum bipartite graph in a triangle-free graph are given in [12].

Even better bounds on the maximum size of bipartite subgraphs are known for 3-regular graphs. We have

- $mc(G) \geq \frac{2}{3}m$ (by the Brooks' theorem),
- $mc(G) \geq \frac{7}{9}m$ for $G \neq K_4$ (see [25],[17]),
- $mc(G) \geq \frac{4}{5}m$ for a triangle-free graph ([15],[5]),
- $mc(G) \geq \frac{6\cdot 2^{k-2}-1}{7\cdot 2^{k-2}-1}m$ if the girth of G is at least $4k+2$ (see [26]).

Locke [18] proved that a $2r$-regular triangle-free graph contains a bipartite graph of $m\frac{r+2}{2r+2}$ edges (and a similar bound for $2r+1$-regular graphs).

We conclude this section with an open problem formulated in [8], which asks for a lower bound on a max-cut expressed with a nonlinear term.

Problem. *Let G be a weighted graph where $w = (w_e)$ is not necessarily nonnegative. Does there exist a constant $c > 0$ such that $mc(G) \geq \frac{1}{2}\sum_{e\in E} w_e + c(\sum_{e\in E} w_e^2)^{1/2}$?*

We conjecture that $c = 1/(2\sqrt{3})$ might be the correct value of the constant c in the above problem. We confirmed in [8] the conjecture with this value of the constant c in a special case when all the entries of w are only ± 1 or 0.

2. Polyhedral upper bounds

There are several polytopes that were studied in the connection with the max-cut problem. Let $G = (V, E)$ be a graph. The *cut polytope* $P_C(G) \subset \mathbb{R}^E$ is defined as the convex hull of the characteristic vectors of the cuts $\delta S, S \subset V$. The *bipartite subgraph* polytope $P_B(G) \subset \mathbb{R}^E$ is defined as the convex hull of the edge sets of all bipartite subgraphs of G. We have $P_C(G) \subset P_B(G)$, and the polytopes are distinct if G has at least two edges. However, $mc(G) = \max\{wx \mid x \in P_B(G)\} = \max\{wx \mid x \in P_C(G)\}$ for a nonnegative edge-weight function w. We prove that the same holds also for their relaxation $Q(G)$ and $S(G)$ which will be defined below.

The bipartite subgraph polytope $P_B(G)$ has been studied in [2]. The following inequalities are known to be facet defining for $P_B(G)$.

$$0 \le x_e \le 1 \qquad \text{for every } e \in E \tag{1}$$

$$x(C) \le |C| - 1 \qquad \text{for every odd cycle } C \tag{2}$$

where $x(C) = \sum_{e \in E} x_e$. Let $Q(G)$ denote the relaxation of $P_B(G)$ defined by inequalities (1) and (2). The graphs for which $P_B(G) = Q(G)$ are called *weakly bipartite* (see e.g.[2], [13]). In [13] it is proved that every graph not contractible to K_5 is weakly bipartite, but a full characterization of weakly bipartite graphs is not known.

The cut polytope $P_C(G)$ was studied e.g. in [4], where it was shown that the following inequalities are valid for $P_C(G)$

$$x(F) - x(C \backslash F) \le |F| - 1 \quad C \text{ a cycle, and } |F| \text{ odd, } F \subset C \tag{3}$$

and an inequality (3) is a facet if and only if C is chordless. Let $S(G)$ denote the relaxation of $P_C(G)$ defined by all inequalities (1) and (3). Barahona and Mahjoub proved that $S(G) = P_C(G)$ if and only if G is not contractible to K_5.

If $G = K_n$ is a complete graph, then the polytope $S(K_n) = M_n$ is called the *metric polytope*. It follows from (3) that the metric polytope is determined by the following system of triangle-inequalities

$$x_{ij} + x_{ik} + x_{jk} \le 2 \qquad \text{for every triple } i, j, k \tag{4}$$

$$x_{ij} - x_{ik} - x_{jk} \le 0 \qquad \text{for every triple } i, j, k \tag{5}$$

There is a great variety of other known facet defining inequalities for the cut polytope, for a survey see e.g. [7]. However, the above listed inequalities (1)–(5) seem to be most useful for practical applications, see e.g. [3]. In this paper we are interested in the following two questions

- what is the relation between the upper bounds on the max-cut obtained from the polytopes $Q(G), S(G)$, and M_n?
- how good are the polyhedral upper bounds?

We show that optimization over all three polytopes gives the same upper bound, which we denote as $\sigma(G)$, and call *a polyhedral upper bound.* We also show that the bound can be very poor for a general graph.

Since $S(G) \subset Q(G)$ by the definition of the polytopes, we have $\max_{x \in S(G)} wx \leq \max_{x \in Q(G)} wx$. By a result of Barahona [1, Remark 5.1], the polytope $S(G)$ is a projection of M_n, and hence $\max_{x \in S(G)} wx = \max_{x \in M_n} wx$.

We need a technical lemma.

Lemma 3. *Let $G = (V, E)$ be a graph and $y = (y_{C,F})$, C cycle, $F \subseteq C$, $|F|$ odd, a nonnegative integer-valued vector such that*

$$\sum_{C,F:F \ni e} y_{C,F} - \sum_{C,F:C \setminus F \ni e} y_{C,F} \geq 0 \tag{6}$$

for every $e \in E$ (summing is over pairs (C,F)). Then there exist a pair of nonnegative integer-valued vectors $\tilde{z} = (\tilde{z}_e), e \in E$, and $\tilde{y} = (\tilde{y}_C), C$ odd cycle, such that

$$\sum_{e \in F} y_{C,F} - \sum_{e \in C \setminus F} y_{C,F} = \tilde{z}_e + \sum_{C \ni e} \tilde{y}_C \tag{7}$$

for every $e \in E$, and

$$\sum_{e \in E} \tilde{z}_e + \sum_{C \text{ odd}} (|C| - 1)\tilde{y}_C \leq \sum_{C,F} (|F| - 1) y_{C,F}. \tag{8}$$

Proof. Let $y = (y_{C,F})$ be given. Set $k = \|y\| = \sum y_{C,F}$, and assume that k is minimum such that the lemma does not hold.

Case (i). Assume that the vector y can be decomposed into a sum $y = y' + y''$, $y', y'' \neq 0$ of two nonnegative integer-valued vectors such that (6) hold for both y' and y''. Then $\|y'\| < k$ and $\|y''\| < k$. Hence there exist a pair $\tilde{y}', \tilde{z}'$ for y', and a pair $\tilde{y}'', \tilde{z}''$ for y'', such that (7) and (8) are satisfied. Then $\tilde{y} = \tilde{y}' + \tilde{y}''$ and $\tilde{z} = \tilde{z}' + \tilde{z}''$ is the required pair of vectors for y.

Case (ii). Assume that the decomposition of case (i) is not possible. Let us introduce an edge vector $a^{C,F} = (a_e^{C,F})$ for every (C, F) defined by

$$a_e^{C,F} = \begin{cases} 1 & \text{if } e \in F \\ -1 & \text{if } e \in C \backslash F \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}$ denote the list of all vectors $a^{C,F}$ where each vector is taken with the multiplicity $y_{C,F}$. Hence $|\mathcal{A}| = k$ where $k = \sum y_{C,F}$. Let $w = \sum_{a \in \mathcal{A}} a$ be the sum of all vectors from $\mathcal{A}$. We have $w \geq 0$ because of (6) since

$$w_e = \sum_{e \in F} y_{C,F} - \sum_{e \in C \backslash F} y_{C,F} \tag{9}$$

for every $e \in E$. We will order the vectors in $\mathcal{A}$ into a sequence $(a_1, \ldots, a_k)$ by the following procedure. Choose a_1 arbitrarily, and assume that $(a_1, \ldots, a_i)$ are already chosen. Set $s_i = a_1 + \ldots + a_i$. Observe that $0 \leq s_i \leq w$ cannot hold, since then a decomposition as in case (i) would be possible. Hence there exists some edge e_i such that either $s_{i,e_i} < 0$ or $s_{i,e_i} > w_{e_i}$. We choose a_{i+1} from $\mathcal{A} \backslash (a_1, \ldots, a_i)$ such that $a_{i+1,e_i} = 1$ in the former case, and $a_{i+1,e_i} = -1$ in the latter case. Such a_{i+1} must exist because $0 \leq w = \sum_{a \in \mathcal{A}} a$.

We claim that

$$\sum_{e \in E} w_e \leq 1 + \sum_{j=1}^{k} (|F_j| - 1). \tag{10}$$

Let (C_i, F_i) denote a pair (C, F) corresponding to a_i. We have, by induction on i, that

$$\sum_{e: s_{i,e} > 0} s_{i,e} \leq 1 + \sum_{j=1}^{i} (|F_j| - 1) \tag{11}$$

(where only the positive entries of $s_{i,e}$ are summed), for every $i = 1, \ldots, k$, since one pair of $+1$ and -1 cancels on the edge e_i every step i when constructing the sequence $(a_1, \ldots, a_k)$. Thus we have proved (10), since it is a special case of (11) with $i = k$.

Set $p := \sum_{e \in E} w_e$, and $q := \sum_{j=1}^{k} (|F_j| - 1)$. Clearly, q is even since each $|F_j|$ is odd, and $p \leq q + 1$ by (10). We distinguish two cases. If p is even, then $p \leq q$ since q is even. We define $\tilde{z}_e := w_e$ for every $e \in E$, and $\tilde{y}_C := 0$ for every odd cycle C. Thus, for every $e \in E$,

$$\sum_{e \in F} y_{C,F} - \sum_{e \in C \setminus F} y_{C,F} = \sum_{i=1}^{k} a_{i,e} = w_e = \tilde{z}_e,$$

and hence (7) is satisfied. We also have

$$\sum_{e \in E} \tilde{z}_e + \sum_{C \text{ odd}} (|C| - 1)\tilde{y}_C = \sum w_e + 0 = p \leq q = \sum_{C,F} (|F| - 1) y_{C,F}$$

by (10). Hence (8) is satisfied as well.

Assume that p is odd. We have $a_i(\delta S)$ even for every $S \subset V$, $i = 1, \ldots, k$, because C_i is a cycle. Hence also $w(\delta S)$ is even for every $S \subset V$, because $w = \sum a_i$ and the property is preserved by taking a sum of vectors. Since p is odd, the support of w must contain an odd elementary cycle, say C_0. Let a_0 denote the characteristic vector of C_0. We define $\tilde{y}$ and $\tilde{z}$ as follows. We set $\tilde{z} := w - a_0$, $\tilde{y}_{C_0} := 1$, and $\tilde{y}_C := 0$ for $C \neq C_0$. We can check that (7) and (8) are satisfied as in the previous case with p even. ■

Theorem 4. *We have* $\max_{x \in S(G)} wx = \max_{x \in Q(G)} wx = \max_{x \in M_n} wx$ *for every graph* G *with nonnegative edge weights* $w = (w_e)$.

Proof. Due to above remarks, it only remains to show that $\max_{x \in Q(G)} wx \leq \max_{x \in S(G)} wx$. Consider the linear program $\max wx$ subject to constraints (1) and (3). Its dual program reads

$$\min \sum_{C,F} (|F| - 1) y_{C,F} + \sum_{e \in E} z_e \tag{12}$$

$$\sum_{e \in F} y_{C,F} - \sum_{e \in C \setminus F} y_{C,F} + z_e \geq w_e \qquad e \in E \tag{13}$$

$$y_{C,F} \geq 0 \qquad C \text{ cycle}, F \subset C, |F| \text{ odd} \tag{14}$$

$$z_e \geq 0 \qquad e \in E \tag{15}$$

where $y_{C,F}$ and z_e are the variables dual to (3) and (1), respectively.

Assume that a basic optimum dual solution is given. We prove that the optimum dual solution can be modified so that $y_{C,F}$ is strictly positive only when $C = F$, i.e. when it is a dual variable of a constraint (2). We may assume, without loss of generality, that the weight function w is rational. Then also the dual variables are rational since we assume that we have a basic solution. Hence, we may assume that $w = (w_e), y_{C,F}$, and z_e are integers. In a way of contradiction assume that $\sum_{e\in E} w_e$ is minimum such that the optimum dual solution cannot be replaced by an *integer* dual solution of the same cost, and using only constraints (2).

Assume that the optimum dual solution was chosen so that, in addition, $\sum y_{C,F}$ is minimum. We claim that $z \leq w$ under this assumption. Indeed, assume that $z_e > w_e$ for some e. Hence there exists some $y_{\bar{C},\bar{F}} > 0$ with $e \in \bar{C}\backslash\bar{F}$, because otherwise the value of z_e can be decreased by 1, which would decrease the cost of the dual solution. We may redefine the dual solution as follows. Set $y'_{\bar{C},\bar{F}} = y_{\bar{C},\bar{F}} - 1$, $z'_e = z_e - 1$ and $z'_f = z_f + 1$ for $f \in \bar{F}$. Thus we found another optimum dual solution with $\sum y'_{C,F}$ smaller, which is a contradiction.

Since $z \leq w$, we have

$$\sum_{e\in F} y_{C,F} - \sum_{e\in C\backslash F} y_{C,F} \geq w_e - z_e \geq 0$$

for every $e \in E$. Thus we may apply Lemma 3 to the vector y. We can easily check that $\tilde{y}$ and $z + \tilde{z}$ present a feasible dual solution to the linear program

$$\max wx \text{ subject to (1) and (2)}$$

This proves the theorem. ■

The following Corollary 5, proved in a direct way in [13], can be derived from a result of [4] by means of Theorem 4.

Corollary 5. ([13]) *The graphs not contractible to K_5 are weakly bipartite.*

Proof. Let G be a graph not contractible to K_5, and w a nonnegative edge-weight function. Then $\max_{x\in S(G)} wx = \max_{x\in P_C(G)} wx$ by [4], and $\max_{x\in S(G)} wx = \max_{x\in Q(G)} wx$ by Theorem 4. We have $\max_{x\in P_B(G)} wx = \max_{x\in P_C(G)} wx$ since w is nonnegative. Hence $\max_{x\in Q(G)} wx = \max_{x\in P_B(G)} wx$, which proves that G is weakly bipartite. ■

Based on Theorem 4, we define $\sigma(G)$, as the common maximum of any of the three optimization problems for a weighted graph G. We have

Corollary 6. *For every $\varepsilon > 0$, there exists a graph G such that $\sigma(G)/mc(G) \geq 2 - \varepsilon$.*

Proof. Let $G = (V, E)$ be a graph of girth at least ℓ. Then the vector $x = (x_e)$ defined by $x_e := \frac{\ell-1}{\ell}$ belongs to the polytope $Q(G)$. Hence $\sigma(G) \geq \frac{\ell-1}{\ell}|E|$. By choosing first ℓ sufficiently large, and then $|V|$ sufficiently large, the statement follows from Theorem 2. ∎

Corollary 6 shows that the polyhedral upper bound $\sigma(G)$ may differ almost by a factor of two from the actual value of the max-cut $mc(G)$, and hence the result of optimization over the metric polytope may be nearly the same as the optimization only over the collection of trivial inequalities $0 \leq x_e \leq 1, e \in E$.

Example. Let $G = (V, E)$ be a graph of odd girth $2\ell + 1$, i.e. no odd cycle of G is shorter than this value. We have already mentioned that it is easy to check that the vector $x = \frac{2\ell}{2\ell+1}(1, \ldots, 1)$ belongs to the polytope $S(G)$. Hence, by the result of Barahona, there exists a vector $\tilde{x} \in M_n$ such that $\tilde{x}_{ij} = \frac{2\ell}{2\ell+1}(1, \ldots, 1)$ for $ij \in E$. We may give an explicit description of the additional entries of $\tilde{x}$ as follows. Let $d(i, j)$ denote the distance of vertices i and j in G. Set

$$\tilde{x}_{ij} = \begin{cases} \min\left(\frac{d(i,j)}{2\ell+1}, \frac{\ell+1}{2\ell+1}\right) & \text{if } d(i,j) \text{ is even} \\ \\ \max\left(\frac{2\ell+1-d(i,j)}{2\ell+1}, \frac{\ell+1}{2\ell+1}\right) & \text{if } d(i,j) \text{ is odd.} \end{cases}$$

It is not difficult to check that $\tilde{x}$ satisfies all triangle inequalities (4) and (5), and hence $\tilde{x}$ belongs to the metric polytope M_n. ∎

3. An eigenvalue upper bound

In this section we recall the definition of an eigenvalue bound studied in [20] and [6], and show that the eigenvalue bound is never worse than a small multiple of the polyhedral bound, but it can be much better in many cases.

Given a weighted graph G with an edge weight function w, the *Laplacian matrix* $L(G, w)$, or simply $L(G)$, of G is defined as follows.

$$L_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \\ \sum_{k=1}^{n} w_{ik} & \text{if } i = j \end{cases}$$

Observe that the diagonal entries of $L(G)$ are the vertex degrees $d_1, \ldots, d_n$ if G is an unweighted graph. In particular, $L(G) = dI - A$ if

G is a d-regular graph, and A is its adjacency matrix. The maximum eigenvalue of a matrix M is denoted by $\lambda_{\max}(M)$.

Lemma 7. ([20]) *We have $mc(G,w) \le \frac{n}{4}\lambda_{\max}(L(G,w))$ for every weighted graph G on n vertices.* ∎

A d-regular graph G is called a *Ramanujan graph* if $\max(|\nu_2|,|\nu_n|) \le 2\sqrt{d-1}$, where $d = \nu_1 \ge \nu_2 \ge \ldots \ge \nu_n$ are the eigenvalues of the adjacency matrix. Hence $\lambda_{\max}(L(G)) \le d + 2\sqrt{d-1}$ for a Ramanujan graph G. A class of Ramanujan graphs with large girth was constructed in [19]. Thus we have

Corollary 8. *The Ramanujan graphs G constructed in [19] satisfy*

(i) (see [20]) $mc(G) \le \frac{1}{4}nd + \frac{1}{2}n\sqrt{d-1}$,

(ii) (see [19]) the girth of G is asymptotically $\frac{4}{3}\log_{d-1} n$. ∎

The upper bound on the max-cut given by Lemma 7 is rather poor for some graphs. For example, it gives the bound $\frac{n^2}{4}$ for the star $K_{1,n}$, while the actual value is $mc(K_{1,n}) = n$. However, the upper bound can be significantly improved by an additional optimization. In [6] we introduced the following upper bound $\varphi(G)$. For a vector $u = (u_1, \ldots, u_n)$, let $\operatorname{diag}(u)$ denote the associated diagonal matrix. We set

$$\varphi(G) = \min \frac{n}{4}\lambda_{\max}(L(G) + \operatorname{diag}(u))$$

where the minimum is taken over all vectors u with $\sum u_i = 0$. We have shown that $mc(G) \le \varphi(G)$ for every weighted graph G. Moreover, the equality $mc(G) = \varphi(G)$ holds for all bipartite graphs (and some other graphs, too).

The value $\varphi(G)$ is computable in polynomial time with arbitrary precision. One of its good properties, which will be used in this paper, is the subadditivity with respect to amalgamation. Let $G = (V,E)$ and $G' = (V',E')$ be a pair of graphs whose vertex sets may intersect but need not be identical. We define the *amalgam* $G + G'$ as the weighted graph on the vertex set $V \cup V'$ with edge weights w'' defined by

$$w''_e = \begin{cases} w_e & \text{for } e \in E \setminus E' \\ w'_e & \text{for } e \in E' \setminus E \\ w_e + w'_e & \text{for } e \in E \cap E' \end{cases}$$

Theorem 9. ([6]) *We have $\varphi(G+G') \leq \varphi(G) + \varphi(G')$ for any pair G and G' of weighted graphs.* ■

We used Theorem 9 to show that $\varphi(G) \leq 1.131mc(G)$ for every planar, or more generally, weakly bipartite graph G with nonnegative edge weights. We can rephrase this result as

Corollary 10. *We have $\varphi(G) \leq 1.131\sigma(G)$ for every graph G with nonnegative edge weights.*

Sketch of the Proof. Let $G = (V, E)$ be a weighted graph with nonnegative edge weights $w = (w_e)$. Let $\sigma(G) = \max\{wx \mid x$ satisfies constraints (1) and (2) $\}$. The dual problem reads

$$\min \sum_{C \text{ odd cycle}} (|C|-1)y_C + \sum_{e \in E} z_e \tag{16}$$

$$\sum_{C \ni e} y_C + z_e \geq w_e \quad (e \in E) \tag{17}$$

$$y \geq 0 \tag{18}$$

$$z \geq 0 \tag{19}$$

The constraint (17) of the dual program says nothing else but that G is a subgraph of a weighted graph obtained by amalgamation of odd cycles C with weight y_C, and edges (as copies of K_2) with weights z_e. Since $\varphi(C_k) = \frac{n}{2}(1+\cos\frac{\pi}{k})$, and $\sigma(C_k) = k-1$ for an odd cycle C_k, we have $\varphi(C_k) \leq 1.131\sigma(C_k)$ where the worst case is C_5. For K_2, we have $\varphi(K_2) = \sigma(K_2) = 1$. Since the inequality $\varphi \leq 1.131\sigma$ is valid for each of the amalgamated pieces, it is also valid for the resulting graph by Theorem 9. ■

In the above proof, it was crucial to work with a dual solution of the linear program (2), because the approach cannot be applied directly to the dual program of (3). The reason is a presence of negative coefficients in the constraints of (3). For example, consider the weighted graph (K_3, w) with weight -1 on two edges, and +1 on one edge of K_3. This weighted graph corresponds to the constraint $x_{12} - x_{13} - x_{23} \leq 0$. We have $mc(K_3, w) = 0$ while $\varphi(K_3, w) = 1/3$, and hence the ratio φ/mc is not bounded.

Let us also remark that the converse of Corollary 10 is not true with a constant less than 1/2, as shown by the Ramanujan graphs.

We conjecture that the eigenvalue bound $\varphi(G)$ provides a good approximation of the max-cut.

Conjecture. *There exists some constant $c < 2$ such that $\varphi(G)/mc(G) < c$ for every nonnegatively weighted graph G.* ∎

The so far worst known ratio of $\varphi(G)/mc(G)$ is $\frac{25+5\sqrt{5}}{32} \approx 1.131$ which is attained for C_5 and some circulant graphs. Computational experiments for various classes of graphs were done jointly with F. Rendl, and are reported in [21]. A typical ratio of φ/mc is about 1.05 for a graph with nonnegative weights.

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Svatopluk Poljak

Department of Applied Mathematics,
Faculty of Mathematics and Physics,
Charles University,
Malostranské náměstí 25,
118 00 Praha 1, Czechoslovakia.