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Covering the Edges of a Graph by ...

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0. Introduction

Let $\mathcal{H}$ be a class of graphs and G a fixed graph. Denote the minimal number of $\mathcal{H}$-subgraphs covering the edges of G by $\operatorname{cov}(G, \mathcal{H})$. The main aim of the present paper is to survey problems and results concerning the behaviour of $\operatorname{cov}(G, \mathcal{H})$ for various classes $\mathcal{H}$.

Whenever $\mathcal{H}$ is not closed under taking subgraphs we also consider $\operatorname{cov}^*(G, \mathcal{H})$, the minimal number of pairwise edge-disjoint $\mathcal{H}$-subgraphs covering the edges of the graph G.

By $\operatorname{cov}(n, \mathcal{H})$ ($\operatorname{cov}^*(n, \mathcal{H})$) we denote the maximum of $\operatorname{cov}(G, \mathcal{H})$ ($\operatorname{cov}^*(G, \mathcal{H})$) for all n-vertex graphs G. We are interested in results concerning $\operatorname{cov}(n, \mathcal{H})$ and $\operatorname{cov}^*(n, \mathcal{H})$ and more generally in estimates that hold for large classes of graphs G (e.g. we do not consider coverings of planar graphs, complements of paths etc).

Various results concerning multicolor Ramsey numbers, graph colorings and designs can be interpreted as results about edge-coverings. We mention those that seem to be the most important from our point of view.

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However our topic started its independent existence with the classical 1966 paper of Erdős, Goodman and Pósa [26] on coverings by complete graphs. The main result of [26] is the following.

Theorem.

$$\mathrm{cov}^{*}\,(n, K_3 \text{ and } K_2) = \left[\frac{n^2}{4}\right].$$

It is of interest to note that the slightly weaker

$$\mathrm{cov}\,(n, K_3 \text{ and } K_2) = \left[\frac{n^2}{4}\right]$$

has already been proved in a forgotten 1941 paper by M. Hall Jr. [42] (in a somewhat disguised form).

The Erdős-Goodman-Pósa Theorem has an amazing number of generalizations and analogues.

For example the maximal number of edges in a triangle-free graph is also $\left[\frac{n^2}{4}\right]$ [54], [73]. This suggests that a natural way of strengthening Turán-type theorems [32] is to prove the corresponding results for coverings. This idea has lead to some of the most general questions and results about graph coverings (see sections 5 and 6).

Our topic abounds in unsolved problems. During the last 10 years certain methods emerged leading at least to approximate solutions of some old ones. In this paper we attempt to give an account of these new developments and of possible future directions of research.

Notation

G will usually denote a finite, simple, undirected graph with $n = v(G)$ vertices and $e(G)$ edges. We denote the maximal degree and the chromatic number of G by $\Delta(G)$ and $\chi(G)$ respectively.

The subgraph induced by a set of vertices X is denoted by $\langle X \rangle$.

We use the usual notations C_t, K_t and $K_{t,r}$ for cycles, complete graphs and complete bipartite graphs.

Topological H-graphs (subdivisions of a graph H) are called TH-graphs.

For $n = t(k-1) + r$, $0 \le r \le k-2$, let $T_{k-1,n}$ denote the complete $(k-1)$ partite graph of n vertices such that r color-classes contain $t+1$ vertices and $k-r-1$ color-classes contain t vertices. These are the so called Turán-graphs [73].

We have

$$e(T_{k-1,n}) = \frac{n(n-t) - r(t+1)}{2}$$

in particular

$$e(T_2, n) = \left[\frac{n^2}{4}\right].$$

1. Even and odd graphs

The central result of this section is about coverings by subgraphs with all degrees even.

Theorem 1.1. [55] *Every bridgeless graph can be covered by at most 3 even subgraphs.*

This is sharp e.g. for cubic graphs which are not 3-edge-colorable.

The above theorem is just a disguised form of Jaeger's 8-flow theorem [45]. Jaeger's original proof relies on a general result of Edmonds [24] on the coverings of matroids (for definitions see [77]).

Theorem 1.2. *Let M be a matroid with rank function r. M can be covered by t independent sets iff for any subset A of M*

$$t \cdot r(A) \ge |A|$$

holds.

This result is used to derive the following.

Theorem 1.3.

(i) Every 3-edge-connected graph can be covered by at most 3 cotrees (complements of trees).

(ii) Every 4-edge-connected graph can be covered by at most 2 cotrees.

1.1. is a straightforward consequence of 1.3. (ii).

There is a useful sharper version of Theorem 1.1. In proving the 6-flow Theorem Seymour [70] showed that a 3-connected graph can always be covered by 3 even subgraphs one of which is a vertex-disjoint union of cycles. Younger [80] gave a different proof of this lemma extending it to arbitrary bridgeless graphs.

There is also a counterpart of Theorem 1.1. about coverings by odd subgraphs (subgraphs with all degrees odd or zero).

Proposition 1.4. *Every graph G can be covered by at most 4 disjoint odd subgraphs.*

Proof. We may suppose that G is connected. Let T be a spanning tree of G.

It is straightforward to show that any forest can be covered by at most 2 disjoint star-forests consisting of odd stars.

If n is even then G has an even number of even vertices $v_1, \ldots, v_{2t}$. Let P_i denote the path between v_{2i-1} and v_{2i} in T for $i = 1, \ldots, t$. Taking the mod 2 sum of the paths P_i we obtain a forest $F \subset T$ such that $G \setminus F$ is an odd graph. Therefore if n is even then G can be covered by at most 3 disjoint odd subgraphs.

If n is odd then let v be an endvertex of T. $G \setminus v$ is a connected graph with an even number of vertices therefore $T \setminus v$ contains a forest F such that $(G \setminus v) \setminus F$ is an odd graph.

If v is an even vertex of G we add the edge of T incident to v to F.

In any case G is the disjoint union of an odd star, a forest and an odd subgraph . Our proposition follows. ■

It is straightforward to see that W_4 the wheel of 4 spokes cannot be covered by less then 4 disjoint odd subgraphs. It would be nice to have an infinite series of such examples.

For not necessarily disjoint coverings we have the following.

Problem 1.5. *Is it true that every graph can be covered by at most 3 odd subgraphs?*

This would be sharp for even graphs with an odd number of vertices.

Trying to generalize Proposition 1.4. one may ask the following. A $1 \pmod k$-coloring of a graph is a coloring where the number of edges of the same color incident to any vertex v of G is either 0 or $\equiv 1 \pmod k$.

What is the minimal number of colors of a $1 \pmod k$-coloring? In particular is it bounded by some function of k?

2. $\mathcal{K}$-free graphs

Suppose $\mathcal{K}$ is a class of graphs and $\mathcal{H}$ consists of those graphs that contain no $K \in \mathcal{K}$ as a subgraph. To determine $\operatorname{cov}(n, \mathcal{H})$ it suffices to consider $\mathcal{K}$-free coverings of K_n.

As noted in [19] $\operatorname{cov}(K_n, \mathcal{H})$ can be thought of as a kind of inverse Ramsey number. Instead of repeating here results about multicolor Ramsey numbers (which can be found e.g. in [35]) we mention a few results that are related to C_3 and C_4-free coverings (which seem to be the most interesting special cases).

If $\mathcal{K}$ consists of all odd cycles then $\mathcal{H}$ is of course the class of bipartite graphs. It is obvious that $\operatorname{cov}(n, \text{bipartite graphs}) = \lceil \log n \rceil$. On a slightly less frivolous level we have the following,

Proposition 2.1. [43]

$$\operatorname{cov}(G, \textit{ bipartite graphs }) = \lceil \log \chi(G) \rceil.$$

If $\mathcal{K}$ consists of the shortest odd cycle then we have the famous result of Schur [69].

Theorem 2.2.

$$c_1 \log n / \log\log n < \operatorname{cov}(n, C_3\textit{-free graphs}) \le c_2 \log n \quad \textit{for some} \quad c_1, c_2 > 0.$$

Here of course the main problem is to determine whether the lower bound is essentially sharp [15], [20]. Still it would be interesting to give some formula which determines $\operatorname{cov}(G, C_3\text{-free graphs})$ up to say a $\log\log n$ factor. (As it is well-known there are triangle-free graphs with arbitrarily high chromatic number and therefore no analogue of Proposition 2.1 can hold).

There is a recent result of Tuza [75] which is rather similar to Theorem 2.2.

Theorem 2.3.

$$c_1 \log n / \log\log n \le \operatorname{cov}(n, \textit{ perfect graphs }) \le \operatorname{cov}^*(n, \textit{ perfect graphs }) \le c_2 \log n$$

for some $c_1, c_2 > 0$.

Again the question is (as suggested by Tuza) whether the lower bound is sharp. (Note that if the famous conjecture of Berge is true then perfect graphs are exactly the graphs containing no odd cycles of length ≥ 5 or their complements as induced subgraphs.)

Let us turn now to C_4-free coverings. The maximal number of edges in a C_4-free graph is about $n^{3/2}$ (see [32]).The following consequence of results due to Graham and Chung [19] is therefore not unexpected.

Theorem 2.4.

$$\operatorname{cov}(n, C_4\textit{-free graphs}) = (1 + o(1))\sqrt{n}.$$

It is natural to ask whether this estimate has some kind of extension to C_4-free coverings of arbitrary graphs. In response to a recent question of Erdős, Füredi [33] proved the following.

Example 2.5. *For n sufficiently large there is a $K_{3,2}$-free graph G of n vertices and $n^{1.36}$ edges such that any C_4-free subgraph of G has at most $n^{4/3}\log n$ edges.*

For such a graph G we have $\operatorname{cov}(G, C_4\text{-free graphs}) \ge n^{0.2}$ even though any x vertices of G span at most $c\,x^{3/2}$ edges for some constant c.

3. Forests

When F is a class of forests it is usually easy to give good estimates for

$$\operatorname{cov}(n, F) = \operatorname{cov}(K_n, F).$$

Moreover in some cases there are nice minimax or "almost minimax" formulae for $\operatorname{cov}(G, F)$. The first of these is the Nash-Williams formula for the arboricity of a graph G [56].

Theorem 3.1.

$$\operatorname{cov}(G,\ \textit{forests}\ ) = \max\left\{\left\lceil\frac{e(H)}{v(H)-1}\right\rceil, H \textit{ is an induced subgraph of } G\right\}.$$

This beautiful result is of course a special case of Theorem 1.2. The classical result of Vizing [76] should also be mentioned here (see also [30], [72]).

Theorem 3.2.

$$\Delta(G) \le \operatorname{cov}(G,\ \textit{matchings}\ ) \le \Delta(G) + 1.$$

Half way between matchings and forests we find the class of linear forests (sets of pairwise vertex disjoint paths). It is conjectured [1] that there is a formula for cov(G, linear forests), the linear arboricity of G, similar to the above ones.

Conjecture 3.3.

$$\lceil\Delta(G)/2\rceil \le \operatorname{cov}(G,\ \textit{linear forests}\ ) \le \left\lceil\frac{\Delta(G)+1}{2}\right\rceil.$$

There is also a "chain" of conjectures [40] which connects Vizing's Theorem and 3.3. A linear k-forest is a graph whose connected components are paths of length at most k.

Conjecture 3.4. *For every $i \ge 1$ we have*

$$\operatorname{cov}(G,\ \textit{linear } i\textit{-forests}\ ) \le \begin{cases} \lceil\Delta(G)\cdot(n+1)/2\cdot\left[\frac{in}{i+1}\right]\rceil & \textit{if } \Delta(G) \ne n-1, \\ \lceil\Delta(G)/2\cdot\left[\frac{in}{i+1}\right]\rceil & \textit{if } \Delta(G) = n-1. \end{cases}$$

This is known to be true e.g. for cubic graphs if $i = 2, 3$ [9]. Conjecture 3.3 has been proved for $\Delta(G) \le 6$, $\Delta(G) = 8$, $\Delta(G) = 10$ (for references see [3]).

More importantly 3.3 is asymptotically true as shown by Alon [3].

Theorem 3.5. *For every $\varepsilon > G$ there is a $\Delta_0 = \Delta_0(\varepsilon)$ such that if G is a graph with $\Delta(G) \geq \Delta_0$ then*

$$\operatorname{cov}(G, \textit{ linear forests }) \leq \left(\frac{1}{2} + \varepsilon\right) \Delta(G).$$

Let us say a few words about Alon's method. By probabilistic methods it is shown in [3] that G can more or less be decomposed into "not too many" subgraphs or large girth and small maximal degree. This reduces the proof of 3.5 to the following statement (which is interesting on its own right).

Theorem 3.6. *Let G be a d-regular graph, d even, with girth $g \geq 50d$. Then*

$$\operatorname{cov}(G, \textit{ linear forests }) = \frac{d}{2} + 1.$$

By a well-known theorem of Petersen [58] such a G can be partitioned into $\frac{d}{2}$ 2-factors $F_1, \ldots, F_{d/2}$. $L(G)$ the line graph of G is a $(2d-2)$-regular graph and the cycles of the 2-factors F_i define a partition of the vertex set of $L(G)$ into sets V_j of size $\geq 50d$. It is shown by using the Lovász Local Lemma that there is an independent set M of $L(G)$ that intersects each of the above sets V_j. M corresponds to a matching of G, the sets $F_i \setminus M$ correspond to linear forests of G and 3.6 follows.

Using a somewhat similar approach Algor and Alon [2] gave an upper bound for the number of star forests covering a graph G which has further been improved by Alon, McDiarmid and Reed [4].

Theorem 3.7.

$$\operatorname{cov}(G, \textit{ star forests }) \leq \frac{\Delta(G)}{2} + c \log \Delta(G) \quad \textit{for some} \quad c > 0.$$

In fact it is proved in [4] that the linear arboricity of a graph G cannot exceed the arboricity of G by more than $c \log \Delta(G)$. On the other hand in [2] some examples are given with $\operatorname{cov}(G, \text{ star forests }) \geq \frac{\Delta(G)}{2} + c' \log \Delta(G)$.

Let us have a look at more general questions. Forests can be defined as graphs not containing TK_3-subgraphs. By Kuratowski's Theorem planar graphs can be defined as graphs not containing TK_5 or $TK_{3,3}$ subgraphs and indeed there is a close relationship between the arboricity of a graph G and between cov(G, planar graphs), the thickness of G.

Proposition 3.8.

$$\operatorname{cov}(G,\ \textit{planar graphs}\ ) \leq 3 \operatorname{cov}(G,\ \textit{forests}\ ).$$

There are many results about the thickness of specific graphs like the following (see [7]).

Theorem 3.9.

$$\operatorname{cov}(K_n,\ \textit{planar graphs}\ ) = \begin{cases} \left[\frac{1}{6}(n+7)\right] & \textit{if} \quad n \neq 9, 10, \\ 3 & \textit{if} \quad n = 9, 10. \end{cases}$$

However there seem to be no really good general estimates for the thickness and related covering invariants.

We would like to suggest the investigation of coverings by Θ-free graphs ($T(K_4 \setminus K_2)$-free graphs). This appears to be the simplest interesting class defined by the exclusion of topological subgraphs.

Question 3.10. *Is there a polynomial-time algorithm which determines* $\operatorname{cov}(G,\ \Theta\textit{-free graphs}\ )$ *up to an additive constant?*

4. Trees

The number of trees covering a connected graph G is of course given by Theorem 3.1. For partitions we have the following result of Chung [16].

Theorem 4.1. *Let G be a connected graph. Then*

$$\operatorname{cov}^*(G,\ \textit{trees}\ ) \leq \left[\frac{n+1}{2}\right].$$

This is sharp for complete graphs.

To obtain further results we have to consider coverings by special types of trees.

Trees of diameter 1 are stars. The minimal number of stars covering a graph G is a much investigated invariant. We just quote one well-known observation of Gallai.

Proposition 4.2. *Let t be the maximal number of independent vertices of G. Then*

$$\operatorname{cov}^*(n,\ stars) = n - t.$$

For coverings by trees of bounded diameter the following results were proved by Lovász and Gallai (see [53]).

Theorem 4.3. *Let G be a connected graph. Then*

(i) $\operatorname{cov}^*(G,\ \textit{trees of diameter} \le 3) \le \left[\frac{2n}{3}\right]$

and for $i \ge 4$.

(ii) $\operatorname{cov}^*(G,\ \textit{trees of diameter} \le i) \le \left[\frac{n+1}{2}\right]$

In [53] connected graphs G are constructed with

$$\operatorname{cov}^*(G,\ \text{trees of diameter} \le 3) \ge \left[\frac{2n-1}{3}\right].$$

Part (ii) is of course sharp for complete graphs.

The following classical conjecture is due to Gallai.

Conjecture 4.4. *Let G be a connected graph. Then*

$$\operatorname{cov}^*(G,\ paths) \le \left[\frac{n+1}{2}\right].$$

Again this conjecture is sharp for complete graphs. Moreover it is sharp for all odd graphs (in a path-partition of an odd graph every vertex is an endvertex of some path).

The following important result of Lovász [53] together with the method of proof is the starting point for many investigations concerning path and cycle coverings. (The method is to technical to be discussed here.)

Theorem 4.5.

$$\operatorname{cov}^*(n,\ paths\ and\ cycles) \le \left[\frac{n}{2}\right].$$

Corollary 4.6.

(i) $\mathrm{cov}^*\,(n,\ paths\,) \le n-1$

(ii) Let G be an odd graph. Then

$$\mathrm{cov}^*\,(G,\ paths\,) = \frac{n}{2}.$$

4.6 (i) has slightly been improved by Donald [D].

Theorem 4.7.

$$\mathrm{cov}^*\,(n,\ paths\,) \le \left\lceil \frac{3}{4}n \right\rceil.$$

Using the method of Lovász one can also prove a slight extension of 4.6 (ii).

Theorem 4.8. [63] *Suppose that each cycle of the graph G contains a vertex of odd degree. Then*

$$\mathrm{cov}^*\,(G,\ paths\,) \le \left\lfloor \frac{n}{2} \right\rfloor.$$

As we have seen there are relatively strong partial results about path-partitions, however to obtain a full proof of Gallai's conjecture seems to be very difficult.

It was suggested by Chung that one should first prove that connected graphs can at least be covered by $\le \left[\frac{n+1}{2}\right]$ paths. We have the following partial result which shows that this at least is asymptotically true.

Theorem 4.9. [63] *Let G be a connected graph. Then*

(i) $\mathrm{cov}\,(G,\ paths\,) \le \frac{n}{2} + c_1\, e\,(G)/n,$

(ii) $\mathrm{cov}\,(G,\ paths\,) \le \frac{n}{2} + c_2\, n^{3/4}$ *for some* $c_1, c_2 > 0.$

5. Cycles and TH-graphs

One of the most beautiful conjectures of graph theory is due to Hajós (see [21], [53]).

Conjecture 5.1. *Let G be an even graph. Then*

$$\text{cov}^*\,(G,\ \textit{cycles}\,) \le \frac{n-1}{2}.$$

At present a proof of 5.1 seems to be entirely out of reach. Even the following weaker conjecture of Erdős and Gallai is still open (see [25]).

Conjecture 5.2.

$$\text{cov}^*\,(n,\ \textit{cycles and } K_2) \le cn \quad \textit{for some} \quad c > 0.$$

There is reason to believe that $c = 3/2$ is the right constant. As noted in [22] and [59] 5.2 would easily follow from 5.1 with $c = 3/2$.
On the other hand already Gallai observed that $\text{cov}^*(K_{n-3,3}$; cycles and $K_2) \ge 4/3(n-3)$ (see in [26]). We improve this as follows.

Example 5.3. [22], [59] *For every $\varepsilon > 0$ there is a graph G with* $\text{cov}^*\,(G,\ \textit{cycles and } K_2) \ge (3/2-\varepsilon)\,n$.

Proof. Let us consider the graph $K_{n-t,t}$ t odd. Any cycle of this graph has at most $2t$ edges. In a covering of $K_{n-t,t}$ by disjoint cycles and edges there are at least $n-t$ edges as each vertex of odd degree is incident to one of them. To cover the remaining edges we need at least $(n-t)\,(t-1)\,/2t$ cycles and edges. Therefore $\text{cov}^*\,(K_{n-t,t}\,,$ cycles and $K_2) \ge (n-t)\left(1+\frac{c-1}{2c}\right)$ and choosing t and n in an appropriate way our statement follows. ∎

It was observed in [26] that the "greedy algorithm" (taking away always the longest cycle) gives a $0\,(n \log n)$ cycle and edge partition. A similar approach gives the following partial result.

Proposition 5.4. [65] *Let G be a class of graphs closed under taking subgraphs such that for some $\varepsilon > 0$ and $c > 0$ and for all $G \in \mathcal{G}$ we have $e\,(G) \le c\,n^{2-\varepsilon}$. Then for all $G \in \mathcal{G}$ we have* $\text{cov}^*\,(G,\ \textit{cycles and } K_2) \le c'\,n$ *for some $c' > 0$.*

It follows that 5.2 holds e.g. for the class of C_4-free or $K_{3,3}$-free graphs.

This partial result suggest the following somewhat technical problem: Prove that for some $\varepsilon > 0$ there exist constants $c_1, c_2 > 0$ such that every graph has a partition into $c_1 n$ cycles and $c_2 n^{2-\varepsilon}$ edges.

Another consequence of Conjecture 5.1 that first appeared as an open problem in the Erdős-Goodman-Pósa paper was settled by the present author [60].

Theorem 5.5.

$$\text{cov}\,(n,\ \textit{cycles and } K_2) \le n - 1.$$

This is of course sharp for trees. Another formulation of 5.5 is that every bridgeless graph can be covered by $n-1$ cycles. Bondy [12] conjectures that this can be improved to $2/3\ (n-1)$ (which would be best possible).

Another related (and somewhat hopeless looking) conjecture of Bondy is the following common generalization of 5.5 and the famous Cycle Double Cover Conjecture [12]: Every bridgeless graph G has a covering by $n-1$ cycles such that each edge is covered by exactly 2 cycles.

Let R be a class of graphs. A covering C of a graph G by its subgraphs is said to have an R-free transversal if it is possible to select distinct edges from elements of C such that the graph formed by these edges has no R-subgraph.

Using some matroid-theoretic machinery it can be shown that 5.5 is equivalent to the following [60].

Theorem 5.5'. *Every graph G has a covering C by cycles and edges such that C has a cycle-free transversal.*

One advantage of this formulation is that 5.5' can be extended to infinite graphs. Indeed it was observed by Jörgensen [46] that 5.5' holds in fact for infinite graphs.

Furthermore it seems to be possible that the rather involved proof of 5.5 can be replaced by a simpler one by proving 5.5' directly.

It would be interesting to decide whether the above results can be extended to some classes of non-graphic matroids. Goddyn suggested the investigation of cographic matroids, more precisely the following.

Problem 5.6. *Is it true that every 3-edge-connected graph G can be covered by $e(G) - n + 1$ minimal cuts and edges?*

Let us say a few words about the proof of Theorem 5.5.

Suppose that G is connected. For some vertex v of G denote the set of vertices at distance i from v by D_i $(1 = 1, \dots, r)$. Consider the graphs $G_i = \langle D_i \cup D_{i-1} \rangle \setminus \langle D_i \rangle$. By the theorem of Lovász G_i can be covered by $v(G_i)/2$ paths and cycles. If P is a path in G_i then as D_i is an independent set in G_i P is the union of a path P' with both endvertices in D_{i-1} and at most 2 edges. It is easy to see that P' can be embedded into a cycle of G.

Therefore the edges of G_i can be covered by $\frac{|D_i|+|D_{i-1}|}{2}$ cycles of G and twice as many edges. A similar statement holds for $\langle D_r \rangle$.

Summing up we obtain that G can be covered by at most $n-1$ cycles and $2(n-1)$ edges. The main part of the proof of 5.5 consists of showing that we can get rid of the extra edges. ■

The above proof was the first to give an $O(n)$ bound. For another one (that also relies on the theorem of Lovász) see [47].

Let us turn now to more general problems.

For a class of graphs $\mathcal{H}$ let $\mathrm{ex}(n, \mathcal{H})$ denote the maximal number of edges in a $\mathcal{H}$-free graph of n vertices.

Yet another version of 5.5 is the following:

$$\mathrm{ex}(n, \text{ cycles }) = \mathrm{cov}(n, \text{ cycles and } K_2).$$

As noted in the introduction the Erdős-Goodman-Pósa Theorem shows that the same holds for coverings by triangles: $\mathrm{ex}(n, K_3) = \mathrm{cov}(n, K_3 \text{ and } K_2)$.

It seems to be very interesting to find other classes $\mathcal{H}$ with

$$\mathrm{ex}(n, \mathcal{H}) = \mathrm{cov}(n, \mathcal{H} \text{ and } K_2).$$

Conjecture 5.7. [47]

$$\mathrm{cov}(n, \textit{ even cycles and } K_2) = \mathrm{ex}(n, \textit{ even cycles}).$$

This would imply $\mathrm{cov}(n, \text{ even cycles and } K_2) \le \frac{3}{2}(n-1)$. Using again the theorem of Lovász it was proved in [47] that at least $\mathrm{cov}(n, \text{ even cycles and } K_2) \le 2(n-1)$ holds.

A cycle of G is exactly a TK_3-subgraph. Therefore the following conjecture is a generalization of 5.5.

Conjecture 5.8.

$$\operatorname{cov}(n,\, TK_p \text{ and } K_2) \;=\; \operatorname{ex}(n,\, TK_p) \;\; \text{for } p \geq 3.$$

Using 5.5 Jörgensen [47] showed that 5.8 holds for p=4. In fact he proved the following [46].

Theorem 5.9. *Let G be a (possibly infinite) graph. Then G has a covering $\mathcal{C}$ by TK_4-subgraphs and edges such that $\mathcal{C}$ has a TK_4-free transversal.*

The following result also supports conjecture 5.8.

Theorem 5.10. [47] *Let H be a connected graph with $v(H) \geq 3$. Then*

$$\operatorname{cov}(n,\, TH\text{-subgraphs and } K_2) \;\leq\; 400\operatorname{ex}(n,\, TH\text{-subgraphs})$$

6. Complete graphs and copies of a fixed graph

The starting point for this section is of course the Erdős-Goodman-Pósa Theorem. This was extended by Bollobás [10] as follows.

Theorem 6.1.

$$\operatorname{cov}^*(n,\, K_r \text{ and } K_2) \;=\; \operatorname{ex}(n,\, K_r)\,.$$

Of course by Turán's Theorem we know that $\operatorname{ex}(n, K_r) = e(T_{r-1,n})$.

For coverings by triangles we have sharper results.

Theorem 6.2. [50] *Suppose that every edge of the connected graph G is contained by some triangle of G, $G \neq K_4$. Then*

$$\operatorname{cov}(G,\, K_3) \;=\; \left[(n-1)^2/4\right].$$

Let $\operatorname{ex}(G, \mathcal{H})$ denote the maximal number of edges of a $\mathcal{H}$-free subgraph of G.

Theorem 6.3.

$$\operatorname{cov}(G, K_3 \text{ and } K_2) \le \operatorname{ex}(G, K_3).$$

This beautiful result is due to Lehel and Tuza [51]. They suggest the following common generalization of 6.2 and the Erdős-Goodman-Pósa Theorem.

Conjecture 6.4.

$$\operatorname{cov}^*(G, K_3 \text{ and } K_2) \le \operatorname{ex}(G, K_3).$$

In fact Lehel and Tuza obtain the following surprisingly general result [51].

Theorem 6.5. *Let H be a non-bipartite graph. Then*

$$\operatorname{cov}(G, H \text{ and } K_2) \le \operatorname{ex}(G, H).$$

Note that by the Erdős-Simonovits Theorem [28] this implies the following.

Corollary 6.6. *Let H be a graph with $\chi(H) = k \ge 3$. Then*

$$\operatorname{cov}(n, H \text{ and } K_2) = \left(1 - \frac{1}{k-1}\right)\binom{n}{2} + o\left(n^2\right).$$

For H bipartite the Erdős-Simonovits Theorem gives us $\operatorname{ex}(n, H) = o\left(n^2\right)$. This shows that for H bipartite 6.5 can not hold and also that

$$\operatorname{cov}^*(n, H \text{ and } K_2) \le \frac{\binom{n}{2}}{e(H)} + o\left(n^2\right)$$

The latter inequality can slightly be improved.

Theorem 6.7. [64] *For every bipartite graph there is a constant $c = c(H) > 0$ such that*

$$\frac{\binom{n}{2}}{e(H)} \le \operatorname{cov}^*(n, H \text{ and } K_2) \le \frac{\binom{n}{2}}{e(H)} + cn.$$

Let us say a few words about the proof of 6.5. Let M be a H-free subgraph of maximal size of the graph G with edge-set E. For an edge $e \in E \setminus M$ $\Gamma(e)$ is defined as the set of edges $e' \in M$ such that there exists a H-subgraph of G containing both e and e'.

By a classical observation of Erdős every graph with m edges contains a bipartite (and therefore H-free) subgraph with more than $\frac{m}{2}$ edges. Using this Lehel and Tuza prove that for every $K \subset E \setminus M$ we have $|\bigcup_{e \in K} \Gamma(e)| \geq |K|$. By the Kőnig-Hall Theorem this implies the existence of an injection $g : E \setminus M \to M$ with $g(e) \in \Gamma(e)$ for each $e \in E \setminus M$. This proves 6.5.

The above proof gives the following slightly stronger result.

Corollary 6.8. [47] *Let $\mathcal{H}$ consist of non-bipartite graphs. Then every graph G has a covering $\mathcal{C}$ by $\mathcal{H}$-subgraphs and edges such that $\mathcal{C}$ has a $\mathcal{H}$-free transversal.*

It would be interesting to know whether 6.8 can be extended to infinite graphs. The following seems to be the most simple case.

Conjecture 6.9. [47] *Every countable graphs has a covering $\mathcal{C}$ by triangles and edges such that $\mathcal{C}$ has a triangle-free transversal.*

It is easy to see that 6.9 holds e.g. for locally finite graphs.

Let us denote the complete r-uniform hypergraph on p vertices by K_p^r. Lehel [49] obtained a generalization of 6.3 for hypergraphs. By extending our notation to hypergraphs in the obvious way we can formulate his result as follows.

Theorem 6.10. *Let M be an r-uniform hypergraph $1 < r < p$. Then*

$$\operatorname{cov}\left(M, K_p^r \text{ and } K_r^r\right) \leq \operatorname{ex}\left(M, K_p^r\right).$$

Corollary 6.11.

$$\operatorname{cov}\left(n, K_p^r \text{ and } K_r^r\right) = \operatorname{ex}\left(n, K_p^r\right).$$

(Here the numbers $\operatorname{ex}(n, K_p^r)$ are just the Turán numbers and for $r \geq 3$ these are not explicitly known.)

The above corollary was conjectured by Bollobás. Erdős and Sauer [25] conjecture that 6.11 holds for partitions instead of coverings and Lehel [49] made the stronger conjecture that in fact 6.10 holds for partitions.

When does a graph G have an H-decomposition (i.e. a covering by disjoint copies of another graph H)? This kind of questions are closely related to questions about coverings.

The most well-known result in this area is Wilson's Theorem [78] about decompositions of complete graphs. A far-reaching generalization of Wilson's result has recently been obtained by Gustavson [36].

Theorem 6.12. *For any graph H there exists an $\varepsilon_H > 0$ and an integer n_H such that if G is a graph satisfying*

(i) $e(G)$ is divisible by $e(H)$

(ii) every degree of G is a non-negative integer linear combination of the degrees of H

(iii) $n > n_H$

(iv) the minimum degree of G is at least $(1 - \varepsilon_H)n$ then G has a H-decomposition.

Of course conditions (i) and (ii) are necessary for G to have a H-decomposition.

Note that in the case of triangle-decompositions 6.12 essentially settles a classical 1970 problem of Nash-Williams [57].

The proof of 6.12 for K_r-decompositions builds upon earlier work of Chetwynd and Häggkvist [14] on the completion of partial latin squares.

From here the general case is obtained basically by applying Wilson's Theorem. The whole proof of this beautiful result takes up about 100 pages.

7. Some other classes

Graham and Pollak [34] proved that to partition K_n we need at least $n-1$ complete bipartite graphs.

This gives us the following.

Theorem 7.1.

$$\mathrm{cov}^*\,(n,\ \textit{complete bipartite graphs}\,) = n-1$$

For coverings we have a slightly different result.

Theorem 7.2.

$$n - c\log n \leq \operatorname{cov}(n,\ \textit{complete bipartite graphs}\ ) \leq n - [\log n] + 1 \ \textit{for some}\ c > 0.$$

Here the upper bound is due to Tuza [74] the lower bound has been proved by Rödl with probabilistic methods [68].

During a lecture on coverings Győri observed that for the classes $\mathcal{H}$ considered usually we have $\operatorname{cov}(n,\mathcal{H}) \leq cn$ or $\operatorname{cov}(n,\mathcal{H}) \geq \varepsilon n^2$. He asked whether there are any natural classes $\mathcal{H}$ with $\operatorname{cov}(n,\mathcal{H})$ asymptotically greater than cn for any $c > 0$ and $\operatorname{cov}(n,\mathcal{H}) = o\left(n^2\right)$. Here we present a few examples of such classes $\mathcal{H}$.

Proposition 7.3.

$$c_1 n \log\log n \leq \operatorname{cov}(n,\ 3 - \textit{regular graphs and}\ K_2) \leq c_2 n \log n \ \textit{for some}\ c_1, c_2 > 0.$$

Proof. It is known that $c_1 n \log\log n \leq \operatorname{ex}(n,$ 3-regular graphs $) \leq c_2\, n \log n$ [61], [66]. The lower bound is immediate. Let r be the maximal number of vertices in a 3-regular subgraph of a graph G. Then we have $e(G) \leq c_2(n-r)\log(n-r) + r(n-r) + \binom{r}{2}$. This gives us $e(G) \leq (c_2+1)n\log n$ for $r \leq \log n$ and $r \geq e(G)/(1+c_2)r$ for $r \geq \log n$. It easily follows that taking away always the maximal 3-regular subgraph we obtain a subgraph of G with $\leq (c_2+1)n\log n$ edges in $0(n\log n)$ steps. This proves the upper bound. ■

It doesn't seem to be so easy to estimate cov(n, connected 3-regular graphs and K_2).

A "kite" is the union of a C_4 and a path having exactly one vertex in common which is an endvertex of the path.

Proposition 7.4.

$$\operatorname{cov}(n,\ \textit{kites and}\ K_2) = \operatorname{ex}(n, C_4) \ \textit{for}\ n > n_0.$$

Proof. Let G be a connected graph containing a C_4 subgraph Q. There are at most $4(n-1)$ edges incident to the vertices of Q. By Theorem 4.5 the subgraph R induced by the remaining $n-4$ vertices can be covered by at most $\frac{n-4}{2}$ paths and cycles. G is connected therefore the union of Q and a cycle or path of R can always be covered by at most 2 kites of G. Therefore G can be covered by at most $5n$ kites and edges.

On the other hand we have $\mathrm{ex}(n, C_4) \geq cn^{3/2}$ for some $c > 0$ (see [32]) and our proposition easily follows. ■

Let Q_k denote the k-dimensional cube.

Proposition 7.5. *For every $\varepsilon_1, \varepsilon_2 > 0$ there exists an $N = N(\varepsilon_1, \varepsilon_2)$ such that for $n \geq N$ we have*

$$n^{2-\varepsilon_1} \leq \mathrm{cov}(n,\ cubes\ ) \leq \varepsilon_2 n^2.$$

Proof. As $\mathrm{ex}(n, Q_k) = o(n^2)$ and Q_k has $2^k \cdot k$ edges it follows that

$$\mathrm{cov}(n, Q_k \text{ and } K_2) \leq \frac{\binom{n}{2}}{2^k \cdot k} + o(n^2).$$

On the other hand by [71, Theorem 8.2] we have $\mathrm{ex}(n, Q_k) \geq c_k\, n^{2-\varepsilon_k}$ where $\varepsilon_k = \frac{2^k - 2}{2^{k-1}(k-1)}$ for some $c_k > 0$. This gives us

$$\mathrm{cov}(n,\ \text{cubes}\ ) \geq \frac{c_k}{(k-1)2^{k-1}}\, n^{2-\varepsilon_k}.$$

Our proposition follows. ■

Similar results hold for say $\mathrm{cov}(n$, symmetric complete bipartite graphs).

In view of these examples we suggest a modified version of Győri's question. For a class of graphs $\mathcal{H}$ we call a graph G $\mathcal{H}$-covered if each edge of G is contained by some $\mathcal{H}$ subgraph. If $\mathcal{H}$ does not contain K_2 then apart from a few cases (like when $\mathcal{H}$ is the class of cycles) $\mathcal{H}$-covered graphs form a rather unnatural looking class. Still it may be interesting to investigate $\mathcal{H}$-coverings of such graphs. We denote the maximum of $\mathrm{cov}(G, \mathcal{H})$ for all n-vertex $\mathcal{H}$-covered graphs by $\mathrm{cov}'(n, \mathcal{H})$.

Question 7.6. *Is there a natural class $\mathcal{H}$ with $K_2 \notin \mathcal{H}$ and $\mathrm{cov}'(n, \mathcal{H}) = n^{2-\varepsilon+o(1)}$ for some $1 > \varepsilon > 0$?*

Of course it is hard to say which classes $\mathcal{H}$ should be considered natural.

8. Weighted coverings

Suppose we are given a weight function $w : \mathcal{H} \to \mathbb{Z}$. The weight of a $\mathcal{H}$-covering of the graph G is the sum of the weights of the covering $\mathcal{H}$-subgraphs. Define $\mathrm{cov}_w(G, \mathcal{H})$ as the minimal weight of a $\mathcal{H}$-covering of G and $\mathrm{cov}_w(n, \mathcal{H})$ as the maximum of $\mathrm{cov}_w(G, \mathcal{H})$ for all n-vertex graphs G. We can define $\mathrm{cov}^*(n, \mathcal{H})$ analogously.

In most investigations concerning weighted coverings the weight of $H \in K$ is simply $v(H)$.

The first result of this section settled a conjecture of Katona and Tarján concerning a weighted version of the Erdős-Goodman-Pósa theorem (with weight function v).

Theorem 8.1.

$$\mathrm{cov}_v^*(n,\ \textit{complete graphs}) = \left\lfloor \frac{n^2}{2} \right\rfloor.$$

This was proved independently by Chung [17] and Győri and Kostochka [38] (see also [48]).

Trying to sharpen 8.1 and 6.1 Győri and Tuza [39] considered coverings by complete graphs of restricted order.

Theorem 8.2. *For every $r \geq 4$ we have*

$$\mathrm{cov}_v^*\,(n,\ K_r\ \textit{and}\ K_2) = 2e(T_{r-1,n}).$$

The above results are sharp for the Turán graphs.

Surprisingly the case $r = 3$ is different . As observed in [39] we have

$$\mathrm{cov}_v^*\,(K_{6k+4}, K_3 \text{ and } K_2) = 2e(T_2, n) + 1$$

Still for $r = 3$ the following substitute of 8.2 may be true [39].

Conjecture 8.3.

$$\mathrm{cov}_v^*\,(n, K_3\ \textit{and}\ K_2) = 2e(T_2, n) + o(n^2)$$

It would also be interesting to decide whether $\mathrm{cov}_v(n, K_3 \text{ and } K_2) \leq \left\lfloor \frac{n^2}{2} \right\rfloor$ holds.

Chung, Erdős and Spencer [18] and independently Tuza [74] considered weighted coverings by complete bipartite graphs.

Theorem 8.4.

$$c_1 \frac{n^2}{\log n} \leq \mathrm{cov}_v^* \, (n, \textit{ complete bipartite graphs}) \leq c_2 \frac{n^2}{\log n} \textit{ for some } c_1, c_2 > 0.$$

This result is related to complexity problems of $0-1$ matrices [74]. Tuza [75] also observed the following.

Theorem 8.5.

$$c_1 n \log n / \log \log n \leq \mathrm{cov}_v \, (n, \textit{ perfect graphs}) \leq c_2 n \log n \textit{ for some } c_1, c_2 > 0.$$

The most beautiful conjecture about weighed coverings is due to Itai and Rodeh [44].

Conjecture 8.6. *Let G be a bridgeless graph. Then*

$$\mathrm{cov}_v \, (G, \textit{ cycles}) \leq e(G) + n - 1.$$

(Note that $\mathrm{cov}_v(G,$ cycles) is just the minimal number of edges in a cycle-covering.)

The above conjecture is closely related to the theory of integer flows. Indeed it is a simple consequence of Theorem 1.3 (ii) that 8.6 holds for 4-edge-connected graphs. By recent work of Alspach, Goddyn and Zhang [6] related to the Cycle Double Cover Conjecture we have the following. If G is a bridgeless graph containing no topological Petersen subgraphs then $\mathrm{cov}_v(G,$ cycles) is exactly the length of the shortest Chinese Postman Tour. Therefore 8.6 also holds for such graphs.

The best general estimate is due to Fraisse [31] (see also [5], [8]).

Theorem 8.7. *Let G be a bridgeless graph. Then*

$$\mathrm{cov}_v(G, \textit{ cycles}) \leq e(G) + \frac{5}{4}(n-1).$$

Finally let us mention a problem about weighted coverings with weight function different from v.

Erdős suggested the investigation of $\mathrm{cov}_{v-1}\,(G,$ complete graphs) where the weight of K_r is $v(K_r) - 1 = r - 1$. Győri [37] conjectures the following.

Conjecture 8.8.

$$\mathrm{cov}_{v-1}(n,\ \textit{complete graphs}) \leq \left\lceil \frac{n^2}{4} \right\rceil .$$

At first sight this conjecture seems to be just a slightly stronger version of Theorem 8.1. However as noted in [37] it is open even in the case of K_4-free graphs.

9. Other directions

There is a way of strengthening certain results about weighted coverings. The following theorem implies the upper bound in 8.4.

Theorem 9.1. [29] *Every graph G has a decomposition $\mathcal{D}$ into complete bipartite subgraphs such that each vertex of G is contained by at most $c\frac{n}{\log n}$ elements of $\mathcal{D}$ for some $c > 0$.*

This is of course sharp as shown by the lower bound in 8.4.

One can consider a similar problem for cycle-coverings. For a bridgeless graph G cd(G), the cycle-depth of G is the smallest number k such that G has a covering $\mathcal{C}$ by cycles such that every vertex of G is covered by at most k elements of $\mathcal{C}$.

By the Seymour-Younger result mentioned after 1.3 we have $\mathrm{cd}(G) \leq \Delta(G) + 1$ for bridgeless graphs G. It should not be to hard to improve this to $\mathrm{cd}(G) \leq \Delta(G)$ (note that this inequality would follow from the truth of the Cycle Double Cover Conjecture).

On the other hand it is easy to see that $\mathrm{cd}(K_{3,n-3}) \geq \frac{3}{2}(n-3)$ and for the Petersen graph P we have $\mathrm{cd}(P) = 3 = \frac{3}{2}\Delta(P) + 1$.

We propose the following.

Conjecture 9.2. *Let G be a bridgeless graph. Then*

$$\mathrm{cd}(G) \leq \frac{2}{3}\Delta(G) + 1.$$

Let us mention a recent result of Li [52] on path double covers that settles a conjecture of Bondy.

Theorem 9.3. *Every graph G has a covering by n paths such that each edge is contained by exactly two paths and each vertex is the endvertex of exactly two paths (we allow paths of length 0).*

Finally we return to our starting point, to coverings by complete graphs.

The following is an old result of Lovász [53].

Theorem 9.4. *Suppose $k=\binom{n}{2}-e(G)$ and t is the greatest natural number such that $t^2-t\le k$. Then*

$$\operatorname{cov}(G,\ \textit{complete graphs})\le k+t.$$

Caen, Erdős, Pullman, Wormald [13] proved the following Nordhaus-Gaddum type results.

Theorem 9.5.

$$\operatorname{cov}(G,\ \textit{complete graphs})\cdot\operatorname{cov}(\overline{G}\ \textit{complete graphs})\le\left(\frac{1}{256}+o(1)\right)n^4.$$

Theorem 9.6.

$$\operatorname{cov}(G,\ \textit{complete graphs})+\operatorname{cov}(G,\ \textit{complete graphs})\le\left(\frac{1}{2}+o(1)\right)n^2.$$

This was conjectured by Brigham, Dutton and Taylor.

Following the suggestion of Erdős the present author improved this as follows.

Theorem 9.7. [62]

$$\operatorname{cov}(G,\ \textit{complete graphs})+\operatorname{cov}(\overline{G},\ \textit{complete graphs})\le\frac{1}{4}n^2+2$$

if n is sufficiently large.

Obviously many generalizations are possible [13], we just mention one additional result.

Theorem 9.8. [64] *Suppose that G_1, G_2 and G_3 form a partition of K_n. Then*

$$\sum_{i=1}^{3} \mathrm{cov}(G_i, \textit{ complete graphs}) \leq \left(\frac{2}{5} + o(1)\right) n^2.$$

Two recent conjectures of Winkler [79] show that the endless flow of problems and results concerning extensions of the Erdős-Goodman-Pósa Theorem continues.

Conjecture 9.9. *If maximal cliques are removed one by one from any n vertex graph, then the graph will be empty after at most $\frac{n^2}{4}$ steps.*

Conjecture 9.10. *If maximal cliques are removed one by one from any n vertex graph then the graph will be empty after the sum of the numbers of vertices has reached at most $\frac{n^2}{2}$.*

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Note added in proof. Recently Sean McGuinness obtained a surprisingly short proof of Conjecture 9.9.

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