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Graphs which do not Embed a Given Graph and the Ramsey Property

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ABSTRACT

For graphs A and B the relation $A \to (B)^1_r$ means that for every r-coloring of the vertices of A there is a monochromatic copy of B in A. Forb(G) is the family of graphs H such that there is no embedding from G into H. A family $\mathcal{F}$ of graphs is Ramsey if for all graphs $B \in \mathcal{F}$ there is a graph $A \in \mathcal{F}$ such that $A \to (B)^1_r$. The only graphs G for which it is not known whether Forb(G) is Ramsey are graphs which have a cutpoint u adjacent to every other vertex except one vertex e and the components of $G-u$ not containing e can in certain ways be embedded into the component containing e. A particular case in point are the graphs G which consist of a star with center u and an additional vertex e adjacent to exactly one endpoint of the star. We will prove that unless such a graph G is the path of length three, Forb(G) does not have the Ramsey property.

1. Introduction

We only consider finite, undirected, simple graphs. If G is a graph, $V(G)$ denotes the set of vertices of G. If $S \subset V(G)$, $G|S$ will denote the graph H with $S = V(H)$ and in which the vertex a is adjacent to the vertex b if and only if a is adjacent to b in G. For $a \in V(G)$, $G-a$ is the graph

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$G|(V(G)-a)$. $\overline{G}$ denotes the complementary graph to G and if $\mathcal{F}$ is a family of graphs then $\overline{\mathcal{F}} = \{\overline{G} : G \in \mathcal{F}\}$. We denote by P_n the path of length n, that is, the path which has n edges.

The one to one function α from the vertices of a graph A to the vertices of a graph B is an *embedding*, if for every two vertices x, y of A the vertices $\alpha(x), \alpha(y)$ of the graph B are adjacent if and only if the vertices x, y are adjacent in A. For graphs A, B and a positve integer r, the relation $A \rightarrow (B)^1_r$ means that whenever Δ is an r-coloring of the vertices of A, then there is an embedding Φ of B into A such that $\Delta \circ \Phi$ is constant. A family $\mathcal{F}$ of graphs is Ramsey if for all graphs $B \in \mathcal{F}$ and all positive integers r there is a graph $A \in \mathcal{F}$ such that $A \rightarrow (B)^1_r$. Forb(G) is the family of graphs H such that G can not be embedded into H. It is known [1] that if G is 2-connected, then Forb(G) is Ramsey. Clearly, Forb(G) is Ramsey if and only if Forb($\overline{G}$) is Ramsey. Hence, if Forb(G) is not Ramsey, neither G nor $\overline{G}$ are 2-connected. Let $\mathcal{M}$ be the set of graphs G having a cutpoint adjacent to every other vertex of G and $\mathcal{K}$ the set of graphs having a cutpoint adjacent to every other vertex of G except one. It is proven in [2] that if neither G nor $\overline{G}$ are 2-connected then $G \in \mathcal{M} \cup \mathcal{K} \cup \overline{\mathcal{M}} \cup \overline{\mathcal{K}}$. Also, if $G \in \mathcal{M}$, then Forb(G) is Ramsey if and only if $G = P_2$. Clearly $P_3 \in \mathcal{K}$ holds and it is known [2], that Forb(P_3) is Ramsey. Let now G be a graph in $\mathcal{K}$ and u be the cutpoint of G which is adjacent to every other vertex of G except one. We denote this vertex not adjacent to u by e. We also denote the connected component of $G - u$ which contains e by K^e, by K the graph $K^e - e$, by K^u the graph $R|(V(K) \cup \{u\})$, by K^{ue} the graph $R|(V(K^e) \cup \{u\})$ and by L the graph $(G - u) - K^e$. We will assume that the connected components of L are $A_0, A_1, \ldots, A_{r-1}$. For $i \in r$, A_i^u is the graph $R|(V(A_i \cup \{u\})$ and L^u the graph $R|(V(L) \cup \{u\})$.

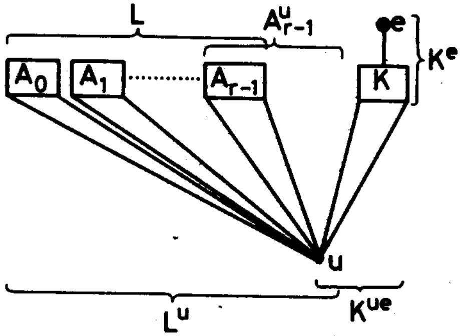

Figure 1.

We will say that G satisfies property $(*)$ if:

$(*)$ (i) there is an $i \in r$ such that A_i does not embed into K and
(ii) there is a $j \in r$ such that A_j^u does not embed into K^e and
(iii) the graph L^u does not embed into the graph K^{ue}.

It is proven in [3] that if G is a graph in $\mathcal{K}$ which satisfies$(*)$, then Forb(G) is not Ramsey. The following graphs $G_r \in \mathcal{K}$ violate the conditions of $(*)$ in points (i) and (ii) and are therefore of particular interest. For $r > 1$, G_r will be the graph with vertex set $V(G) = \{u, e\} \cup A$ where $A = \{a_i : i \in r+1\}$. The edge set $E(G) = \{\{u, a_i\} : i \in r+1\} \cup \{\{e, a_r\}\}$. Note that if we allow $r = 1$ in the definition of G_r, we get $G_1 \cong P_3$. That Forb(P_3) is Ramsey has been shown in [2].

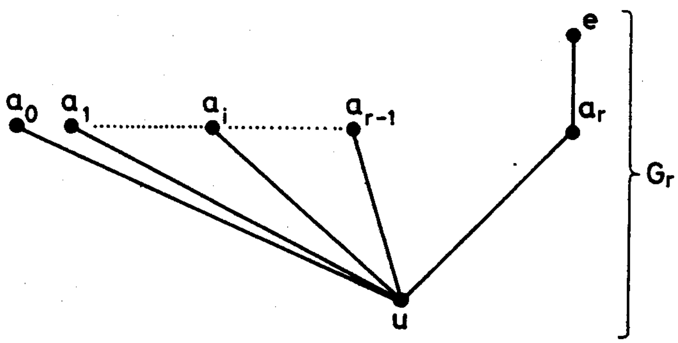

Figure 2.

We will prove that for all $r > 1$, Forb(G_r) is not Ramsey. The proof of this result shares with [2] and [3] the idea that if Forb(G) is Ramsey, then Forb(G) has the so called *disjoint amalgamation property*.

If $\mathcal{F}$ is a family of graphs we say that $\mathcal{F}$ has *amalgamation* if

$$\forall A, B \in \mathcal{F}, a \in V(A), b \in V(B)$$

$\exists C \in \mathcal{F}$ and embeddings $\phi : A \mapsto C, \psi : B \mapsto C$ such that $\phi(a) = \psi(b)$.

If in addition $(\phi(A) - \phi(a)) \cap (\psi(B) - \psi(b)) = \emptyset$, we say that $\mathcal{F}$ has *disjoint amalgamation*. The family $\mathcal{F}$ of graphs has *join-embedding* if

$\forall A, B \in \mathcal{F}\ \exists C \in \mathcal{F}$ and embeddings $\phi : A \mapsto C, \psi : B \mapsto C$.

We will use the following two results from [2].

Lemma 1. [2] *For any graph G,* Forb(G) *has join-embedding.*

Theorem 1. [2] *If the family of graphs $\mathcal{F}$ is Ramsey and has join-embedding, then $\mathcal{F}$ has disjoint amalgamation.*

2. Proof of the theorem

Theorem 2. *The family* Forb(G_r) *of graphs is not Ramsey unless $r = 1$.*

Proof. Let Q_n be the graph with vertex set $V(Q_n) = \{v, w, d\} \cup \{b_i : i \in n\}$ and edge set $E(Q_n) = \{d, w\} \cup \{\{v, b_i\} : i \in n\}\} \cup \{\{w, b_i\} : i \in n\}\}$. Let P_n be the path of length $n - 1$ with vertex set $\{p_0, p_1, \dots, p_{n-1}\}$.

Put now $n = 6r + 3$ and $m = 6r^2 - 3r$. Clearly, $P_n \in$ Forb(G_r) and $Q_m \in$ Forb(G_r). We will prove that there is no graph $R \in$ Forb(G_r) and embeddings ϕ and ψ from P_n and Q_m into R respectively such that $\phi(p_0) = \psi(v)$ and $V(\phi(P_n - p_0)) \cap V(\psi(Q_m - v)) = \emptyset$. Hence we will have shown that Forb(G_r) does not have disjoint amalgamation. Lemma 1 and Theorem 1 imply then that Forb(G_r) is not Ramsey.

Assume to the contrary that there is a graph $R \in$ Forb(G_r) and embeddings ϕ and ψ from P_n and Q_m into R respectively such that $\phi(p_0) = \psi(v)$ and $V(\phi(P_n - p_0)) \cap V(\psi(Q_m - v)) = \emptyset$. Of course we may assume without loss of generality that the embeddings ϕ and ψ are the identity on the graphs P_n and Q_m respectively. For $c \in V(Q_m - v)$ denote by $N(c)$ the set $\{p_j : j \in n$ and c is adjacent to $p_j\}$ and let $N(p_j) = \{c \in V(Q_m - v) : c$ is adjacent to $p_j\}$.

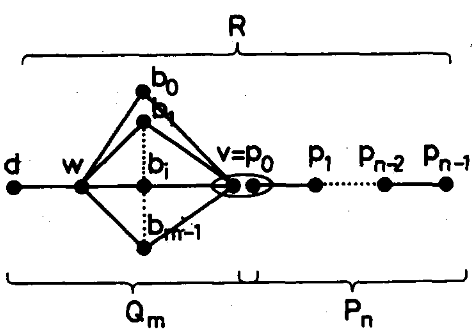

Figure 3.

Lemma 2. *If for some vertex $c \in V(Q_m - v)$, $|N(c)| \geq 2r + 3$ then $N(c) = V(P_n)$.*

Proof. Let $c \in V(Q_m - v)$ with $|N(c)| \geq 2r + 3$. Assume to the contrary that there is a vertex p_{j_0} which is not adjacent to c, but that p with either $p = p_{j_0 - 1}$ or $p = p_{j_0 + 1}$, is adjacent to c. Then there are at least $2r$ vertices $S \subseteq N(c)$ which are neither equal to p_{j_0} nor adjacent to p or p_{j_0}. Choose a subset $T \subset S$ such that no two vertices of T are adjacent. Then $R|(T \cup \{p_{j_0 - 1}, p, c\})$ is a copy of G_r. ■

Corollary. *$|N(w)| \leq 2r + 2$ and $|N(d)| \leq 2r + 2$.*

Proof. If say $|N(w)| \geq 2r + 3$ holds, then by the previous Lemma, the vertex w has to be adjacent to all of the vertices of the path P_n. But because Q_m is embedded in R, neither w nor d is adjacent to $v = p_0$. ■

Lemma 3. *There is an $i \in m$ such that $N(b_i) = V(P_n)$.*

Proof. For $i \in m$ let $\gamma(i) \in n$ be the largest number such that for all $j < \gamma(i)$ either p_j or p_{j+1} is adjacent to b_i. Observe that if $\gamma(i) < n - 1$ then b_i is not adjacent to $p_{\gamma(i)}$ and not adjacent to $p_{\gamma(i)+1}$ but adjacent to $p_{\gamma(i)-1}$. Remember here that for all $i \in m$, the vertex b_i is adjacent to the vertex $p_0 = v$. Hence for each $j < n - 1$, $|\{i : \gamma(i) = j\}| < r$ holds, for otherwise $R|(\{p_j, p_{j+1}\} \cup \{i : b_i \text{ is adjacent to } p_j\})$ would be a copy of G_r in R. The number of numbers $i \in m$ with $\gamma(i) < n - 1$ is therefore at most $(n - 1) \cdot (r - 1) = 6r^2 - 3r - 3$. But $m = 6r^2 - 3r$ and so we may assume without loss of generality that $\gamma(0) \geq n - 2$. Then $|N(b_0)| \geq 2r + 3$ and we get from Lemma 2 that $N(b_0) = V(P_n)$. ■

Observe that if for some subset $S \subseteq V(P_n)$ of pairwise not adjacent vertices no vertex in S is adjacent to w or d and $|S| = r$, $R|(\{d, w, b_0\} \cup S)$ is a copy of G_r in R. Hence for every such set $S \subseteq V(P_n)$ of pairwise not adjacent vertices and $|S| \geq r$ there is a vertex in S which is adjacent to w or d. This is only possible if the number of vertices of P_n which are not adjacent to w and d is at most $2(r-1)$. Then either $|N(w)| \geq \frac{1}{2}(n - 2(r-1)) = 2r+3$ or $|N(d)| \geq 2r + 3$ in contradiction to the Corollary. ■

References

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