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Sparse Random Graphs: A Continuum of Complete Theories

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1. The axioms

We shall described a continuum of countable theories T^α in the first order theory of graphs. This language contains equality and a binary predicate adjacency, written $x \sim y$ assumed reflexive and symmetric. Completeness is proven in Chapter 2. Consistency, by giving a countable model, is given in Chapter 3. The motivation for the definitions of T^α comes from the study of random graphs. This is described in the Remarks which act as a continue. Henceforth let α be a fixed irrational number with $0 < \alpha < 1$. We write T for T^α. The description of T will consist of the Nonexistence Schema and the Generic Extension Schema, defined below. For clearer exposition we also give the Null Closure Schema, an equivalent from of the Nonexistence Schema, and the Extension Schema, a basic case of the Generic Extension Schema.

Remark. The random graph $G(n,p)$ is, formally, a probability space over the set of graph on vertex set $\{1,\ldots,n\}$ where for each pair i,j $\Pr[i \sim j] = p$ and these events and mutually independent. In [1], with S. Shelah, we showed that for any sentence A

$$\lim_{n\to\infty} \Pr[G(n,n^{-\alpha}) \models A] = 0 \text{ or } 1$$

The theorems of T will be precisely those A for which the above limit is one. Such A will be said to hold almost surely (a. s.) in $G(n,p)$ with $p = n^{-\alpha}$.

We call a graph H with v vertices and e edges dense if $v - \alpha e < 0$. We begin the description of T.

Nonexistence schema. For every dense H

"H is not a subgraph"

More formally, let H have vertices $b_1, \ldots, b_v$. The sentence in our schema is

$$\neg(\exists x_1 \ldots x_v) \bigwedge_{i<j} x_i \neq x_j \;\wedge\; \bigwedge_{b_i \sim b_j} x_i \sim x_j$$

Remark. In $G(n,p)$ with $p = n^{-\alpha}$ there are $\sim n^v$ distinct v-tuples $x_1, \ldots, x_v$ and each has probability p^e of having the required edges so that the expected number of $x_1, \ldots, x_v$ giving a subgraph H is $\sim n^v p^e = n^{v-\alpha e}$. With H dense this is o(1) so that all sentences in the Nonexistence Schema hold a.s.

Example. We shall take

$$\alpha = \frac{-1+\sqrt{5}}{2} \sim .618$$

for all examples. Let H be given by Figure 1. Then $v = 5, e = 9$ and $v - \alpha e < 0$.

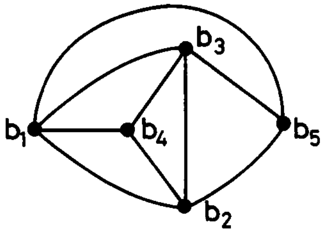


Figure 1.

The remaining axioms involve "extension statements". A rooted graph is a pair (R, H) consisting of a graph H with vertex set $V = V(H)$ and a specified subset $R \subset V$ of vertices, called the roots. We label $R = \{a_1, \ldots, a_r\}$ and $V = R \cup \{b_1, \ldots, b_v\}$. In a graph G we say $y_1, \ldots, y_v$ from an (R, H) extension of $x_1 \ldots, x_r$ if the x's and y's are all distinct and

$a_i \sim b_j$ in H implies $x_j \sim y_j$ in G and olso $b_j \sim b_k$ in H implies $y_j \sim y_k$ in G. The extension statement $\text{Ext}(R, H)$ is that for all distinct $x_1, \ldots, x_r$ there exist an (R, H) extension $y_1, \ldots, y_v$. More formally

$$Ext(R,H) : \forall x_1, \ldots, x_r \bigwedge_{i<j} x_i \neq x_j \Longrightarrow \exists y_1, \ldots, y_v$$

$$\bigwedge_{i,j} x_i \neq y_j \wedge \bigwedge_{j<k} y_j \neq y_k \wedge \bigwedge_{a_i \sim b_j} x_i \sim y_j \wedge \bigwedge_{b_j \sim b_k} y_j \sim y_k$$

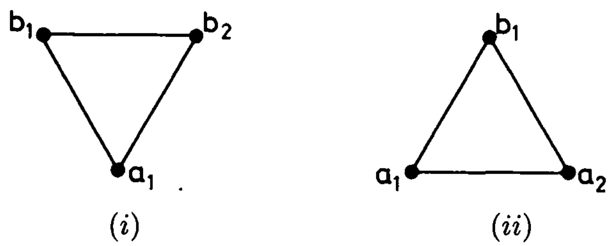


Figure 2.

Example. For Figure 2(i) $\text{Ext}(R, H)$ is that every vertex lies in a triangle while for Figure 2(*ii*) $\text{Ext}(R, H)$ is that every two vertices have a common neighbour. Note that the edges between roots do not affect the sentence $\text{Ext}(R, H)$.

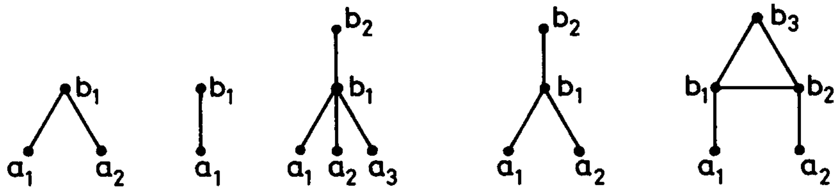
A rooted graph (R, H) is said to have type (v, e) where v is the number of nonroot vertices and e is the number of edges of H, not counting edges between roots.

Example. Figure 2(i) has type (2,3) while Figure 2(ii) has type (1,2).

We call (R, H) sparse if $v - \alpha e > 0$ and dense if $v - \alpha e < 0$. The irrationality of α makes this a sharp dichotomy. We call (R, H) rigid if (S, H) is dense for all S with $R \subseteq S \subset V$. We call (R, H) safe if $(R, H|_S)$ is sparse for all S with $R \subset S \subseteq V$. Note that rigid implies dense and safe implies sparse.

Example. (Recall $\alpha \sim .618$). In Figure 3(iii) is not rigid as with $S = \{a_1, a_2, a_3, b_1\}$ (S, H) has type (1,1) and $1 - \alpha 1 > 0$. Similarly (vi) is not safe as with $S = \{a_1, a_2, b_1\}$ $(R, H|_S)$ has type (1,2) and $1 - 2\alpha < 0$.

We shall require some technical facts. When $R \subset S \subseteq V$ and a graph H on V is given or understood then we shall write (R, S) as a shorthand for $(R, H|_S)$.



(i) Rigid — v=1, e=2

(ii) Safe — v=e=1

(iii) Dense Not Rigid — v=2, e=4

(iv) Sparse Not Safe — v=2, e=3

(v) Rigid — v=3, e=5

Figure 3.

Let $R \subset S \subset V$ and let H be graph on V. Let (R, H) have type (v, e). We may split (R, H) into (R, S) and (S, H) of types, say, (v_1, e_1) and (v_2, e_2) respectively. Note that

$$e = e_1 + e_2$$

$$v = v_1 + v_2$$

This implies: If (R, S) and (S, H) are both dense then (R, H) is dense. If (R, S) and (S, H) and both sparse then (R, H) is sparse.

Let (R, H) be dense, H a graph on V. Let S, $R \subset S \subseteq V$ be minimal with (R, S) dense. For any T with $R \subset T \subset S$ minimality gives (R, T) is sparse and hence (T, S) is dense. Thus (R, S) is rigid. That is, every dense (R, H) contains a rigid "subextension" (R, S).

Let $R \subset S \subset V$, H a graph on V. Assume (R, S) and (S, H) are rigid. We claim (R, H) is rigid. Let $R \subseteq T \subset V$. As (R, S) is rigid $(S \cap T, S)$ is dense. Then $(T, S \cup T)$ is dense as adding roots $T - S$ can only increase e and v remains the same. As (S, H) is rigid $(S \cup T, H)$ is dense. Thus (T, H) is dense, giving the claim.

We continue with the description of the axioms of T.

Extension schema. For every safe (R, H)

$$\text{Ext}(R, H)$$

Remark. In [2] we shoved that $\text{Ext}(R, H)$ holds a.s. in $G(n, p)$ with $p = n^{-\alpha}$ if and only if (R, H) is safe. We indicate the "only if" argument. Given $x_1, \ldots, x_r$ $G(n, p)$ any particular $y_1, \ldots y_v$ give an (R, H) extension with probability p^e so the expected number of such extension is $\sim n^v p^e = n^{v-\alpha e}$. When (R,H) is not sparse it is dense, this quantity is o(1), and $\text{Ext}(R,H)$ a.s. fails. Further, if $\text{Ext}(R,H)$ then all the "subextension" properties $\text{Ext}(R,S)$ must hold so all (R,S) must be sparse and so (R,H) must be safe.

Remark. For any irrational α with $0 < \alpha < 1$ there is a minimal integer s with $\alpha < s/(s+1)$. The rooted graph (R,H) with $R = \{a_1, a_2\}$, $V = R \cup \{b_1, \ldots, b_s\}$ consisting of a part on (in order) $a_1, b_1, \ldots b_s, a_2$ is safe. Here $\text{Ext}(R,H)$ is that every two vertices may be joined by a path of length $s+1$. This implies, in particular, that all models of T are connected graph.

Now for a crucial definition. The t-closure of $x_1, \ldots, x_r$ — denoted $\text{cl}_t(x_1, \ldots, x_r)$ — consists of $x_1, \ldots, x_r$ (this default value) and the union of all $y_1, \ldots, y_v$ with $v \le t$ that from an (R,H) extension of $x_1, \ldots, x_r$ where (R,H) is rigid.

Example. $\text{cl}_1(x_1, x_2)$ consists of x_1, x_2 and all common neighbors y. $\text{cl}_3(x_1, x_2)$ has these points and all y_1, y_2, y_3 giving the (R,H) extension of Figure 3(v) as well as all y_1 or y_1, y_2 or y_1, y_2, y_3 forming any other rigid (R,H) extension over x_1, x_2.

As there are only a finite number of rigid (R,H) with R having r vertices and H having at most $r+t$ vertices the statement $y \in \text{cl}_t(x_1, \ldots, x_r)$ is first order definable.

Remark. In $G(n,p)$ with $p = n^{-\alpha}$ if $x_1, \ldots, x_r$ are chosen at random then a.s. $\text{cl}_t(x_1, \ldots, x_r) = \{x_1, \ldots, x_r\}$. For any rigid (R,H) of type (v,e) the expected number of $y_1, \ldots, y_r$ forming an (R,H) extension is $n^v p^e = n^{v-\alpha e} = o(1)$. However, some $\text{cl}_t(x_1, \ldots, x_r)$ will be nontrivial. With α as in the examples take any y and let x_1, x_2 be two of its neighbours — then $y \in \text{cl}_1(x_1, x_2)$. The sets $\text{cl}_t(x_1, \ldots, x_2)$ give a gradated notion of those y "special" with respect to $x_1, \ldots, x_r$.

The case $r = 0$, $R = \emptyset$ deserves special attention. A rooted graph $(\emptyset, H)$ is dense if H has v vertices, e edges and $v - \alpha e < 0$. It is rigid if $v' - \alpha e' < 0$ for all subgraphs H' with v' vertices, e' edges. The t-closure $\text{cl}_t(\emptyset)$ is the union of all H with $(\emptyset, H)$ rigid, H having at most t vertices. The Nonexistence Schema implies there are no such H (even with H dense) so that $\text{cl}_t(\emptyset) = \emptyset$. Inversely if there exists H on V with $(\emptyset, H)$ dense then we have shown there is a rigid subextension $(\emptyset, S)$ so that $\text{cl}_t(\emptyset) \neq \emptyset$ where $t = |S|$. Thus the Nonexistence Schema is equivalent to the following.

Null closure schema. For all t

$$\mathrm{cl}_t(\emptyset) = \emptyset$$

A rooted graph $(\emptyset, H)$ is safe if $v' - \alpha e' > 0$ for all subgraph H' with v' vertices, e' edges. For such H the Extension Schema gives as a theorem of T that H is a subgraph. When $0 < \alpha < \beta < 1$ are both irrational appropriate H may be used to show that the theories T^α and T^β are distinct.

Definition. $y_1, \ldots, y_v$ is t-generic over $x_1, \ldots, x_r$ if

$$\mathrm{cl}_t(x_1, \ldots, x_r, y_1, \ldots, y_v) = \mathrm{cl}_t(x_1, \ldots, x_r) \cup \{y_1, \ldots, y_v\}$$

Example. y is l-generic over x if rhey have no common neighbours.

As membership in a t-closure is first order definable, so is the notion of being t-generic. Let (R, H) be a rooted graph with $R = \{a_1, \ldots, a_r\}$, $V = V(H) = R \cup \{b_1, \ldots, b_v\}$ as usual. We write $\mathrm{Ext}^t(R, H)$ for the statement that for all distinct $x_1, \ldots, x_r$ there exist $y_1, \ldots, y_v$, distinct from each other and from the $x's$, so that

(i) $a_i \sim b_j$ if and only if $x_i \sim y_j$

(ii) $b_i \sim b_k$ if and only if $y_i \sim y_k$

(iii) $y_1, \ldots, y_v$ is t-generic over $x_1, \ldots, x_r$.

This is a first order sentence. Note that (i), (ii) are stronger than the analogous condition for $\mathrm{Ext}(R, H)$ since we require precisely the edges of H, excluding edges between the x's. We complete the description of T.

Generic extension schema. For all safe (R, H) and all t

$$\mathrm{Ext}^t(R, H)$$

Example. Take (R, H) as in Figure 3(ii) and $t = 1$. For our α we always have $\mathrm{cl}_1(x) = \{x\}$ since there are no rigid graphs with one root and one nonroot. $\mathrm{cl}_1(x, y)$ contains x, y and common neighbours z. Thus $\mathrm{Ext}^1(R, H)$ is that every x has a neighbour y with no common neighbour z. Equivalently, every vertex lies in an edge which is not part of a triangle.

We say that $y_1, \ldots, y_v$ is t-generic over $x_1, \ldots, x_r$ if whenever $z_1, \ldots, z_s$ with $s \leq t$ from a rigid extension over $x_1, \ldots, x_r$, $y_1, \ldots, y_v$ we have no adjacencies between the y_1 and the z_k. This definition allows edges between the x's and the y's (and in use that will be the case) and it allows the x's to have rigid extensions. It prohibits any extension over the x's and y's that actually uses the y's as roots. It implies that the t-closure of $x_1, \ldots, x_r$, $y_1, \ldots, y_v$ equals the t-closure of $x_1, \ldots, x_r$, union $y_1, \ldots, y_v$.

Example. y is l-generic over x if they have no common neighbours.

2. The completeness

Here we prove that the theory T consisting of the Nonexistence Schema and the Generic Extension Schema is complete.

Finite Closure Lemma. For all k, t there exists K so that

$$\vdash_T |\text{cl}_t(x_1, \ldots, x_r)| \leq K + r$$

Proof. There are only a finite number of types (v, e) of dense extensions over r roots with $v \leq t$ and each has $v - \alpha e$ negative. Fix ε positive so that all such $v - \alpha e \leq -\varepsilon$. Let L be such that $r - L\varepsilon < 0$. Set $K = (L-1)t$.

If $|\text{cl}_t(x_1, \ldots, x_r)| > K + r$ there would exist sets $Y_1, \ldots, Y_L$, each of size at most t and disjoint from $R = \{x_1, \ldots, x_r\}$ and each forming a rigid (R, H) extension of R. While the Y_i are not necessarily disjoint we may order them so that each Y_i has at least one element not in $Y_1 \cup \ldots \cup Y_{i-1}$. Set $Y^0 = R$, $Y^i = R \cup Y_1 \cup \ldots \cup Y_i$. As $(R, R \cup Y_i)$ is rigid, $((R \cup Y_i) \cap Y^{i-1}, R \cup Y_i)$ is dense and so (Y^{i-1}, Y^i) is dense. Let (Y^{i-1}, Y^i) have type (v_i, e_i). Then $v_i - \alpha e_i \leq -\varepsilon$. Then Y^L would have $r + v_1 + \ldots + v_L$ vertices and at least $e_1 + \ldots + e_L$ edges but

$$\begin{aligned} r + v_1 + \ldots + v_L - (e_1 + \ldots + e_L) &= r + \sum_{i=1}^{L} v_i - \alpha e_i \\ &\leq r + \sum_{i=1}^{L} (-\varepsilon) = r - L\varepsilon < 0 \end{aligned}$$

which would contradict the Nonexistence Schema.

To prove this in T assume there are distinct $y_1, \ldots, y_{K+1}$, all distinct from $x_1, \ldots, x_r$ and lying on a rigid extension. There are only a finite number of choices for the rigid extension. There are only a finite number of choices for the rigid extension and a finite number of choises for the varieties of overlaps of the extensions and each of these possibilities is ruled out by the Nonexistence Schema.

Example. $\vdash_T |\text{cl}_3(x_1, x_2)| \leq 68$. The extreme case is x_1, x_2 plus twentytwo triples $y_1^i, y_2^i . y_3^i$ each giving the (R, H) extension of Figure 3(v). That graph has 2+22(3)=68 vertices and 5(22)=110 edges and 68-110α = .016... > 0. Indeed one can show that this graph is safe over the empty set so that

$$\vdash_T (\exists x_1, x_2)|\text{cl}_3(x_1, x_2)| = 68$$

Remark. When r, t are fixed and K is considered a function of α we may have K approach infinity as α approaches an appropriate rational. For example, if $0 > 3 - 5\alpha > 1/s$ the above example shows that we must have $K \geq 3s$.

In considering possible t-closures of r vertices we consider graphs H to have labels $x_1, \ldots, x_r$ placed on (possibly equal) vertices. Then $\mathrm{cl}_t(x_1, \ldots, x_r) \cong H$ is first order definable. With H as in Figure 4 it is that there exist y_1, y_2, y_3, y_4 so that $x_1, x_2, y_1, y_2, y_3, y_4$ are distinct, that $\mathrm{cl}_3(x_1, x_2)$ — which we recall is first order definable — has no other elements, and that is adjacencies of these six vertices are precisely as given in the Figure.

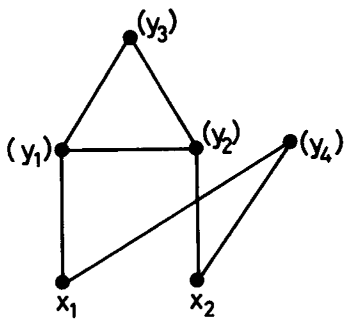


Fig. 4. $\mathrm{cl}_3(x_1, x_2)$. *(The y_i are added for convenience)*

Key Lemma. Let t, r be fixed. Let K be given by the Finite Closure Lemma. Let $s = K + t$. Then for any possible value H' of $\mathrm{cl}_s(x_1, \ldots, x_r)$ and any possible value H of $\mathrm{cl}_t(x_1, \ldots, x_r, y)$ either

$$\vdash_T \mathrm{cl}_s(x_1, \ldots, x_r) \cong H' \Rightarrow (\exists y)\mathrm{cl}_t(x_1, \ldots, x_r, y) \cong H$$

or

$$\vdash_T \mathrm{cl}_s(x_1, \ldots, x_r) \cong H' \Rightarrow \neg(\exists y)\mathrm{cl}_t(x_1, \ldots, x_r, y) \cong H$$

The proof splits into two cases. We call a given H inside if there is a subgraph $H^* \subset H$ with $x_i, \ldots, x_r, y \in H^*$ and $(\{x_1, \ldots, x_r\}, H^*)$ is rigid. Otherwise we call H outside.

Figure 5 illustrates the inside case as y, z_1, z_2 gives a rigid extension of x_1, x_2. By the Finite Closure Lemma H has at most $r + K$ vertices and so $y \in \mathrm{cl}_K(x_1, \ldots, x_r)$. We claim

$$\mathrm{cl}_t(x_1, \ldots, x_r, y) \subset \mathrm{cl}_{K+t}(x_1, \ldots x_r)$$

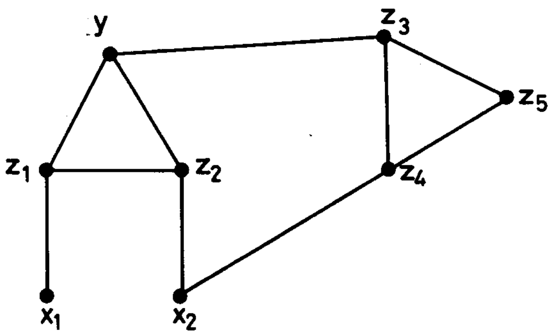


Fig. 5. $H \cong lc_3(x_1, x_2, y)$ *inside*

Set $R = \{x_1, \ldots, x_r\}$. Let $Y = \{y_1, \ldots, y_v\}$ with $y = y_1$, $v \leq K$ and $(R, R \cup Y)$ rigid. Any $z_1 \in \mathrm{cl}_t(x_1, \ldots x_r, y)$ is part of some $Z = \{z_1, \ldots, z_p\}$ with $p \leq t$ and $(R \cup \{y\}, R \cup \{y\} \cup Z)$ rigid. Then $(R \cup \{y\} \cup (z \cap Y), R \cup \{y\} \cup Z)$ is rigid. Adding roots $Y - Z - \{y\}$, $(R \cup Y, R \cup Y \cup Z)$ is rigid. Then $(R, R \cup Y \cup Z)$ is the rigid extension of a rigid extension so is rigid. As $|Y \cup Z| \leq K + t$, $z \in \mathrm{cl}_{K+t}(x_1, \ldots, x_r)$ as claimed.

Now fix $H' \cong \mathrm{cl}_{K+t}(x_1, \ldots, x_r)$. For each $y \in \mathrm{cl}_K(x_1, \ldots, x_r)$ the value $\mathrm{cl}_t(x_1, \ldots, x_r, y)$ is determined as the union of appropriate rigid extensions inside H'. Thus the veracity of $(\exists y)\mathrm{cl}_t(x_1, \ldots, x_r, y) \cong H$ is determined. As H' is finite this is a finite procedure, provable in T.

Now assume H is outside, as in Figure 6. Let H^- be $x_1, \ldots, x_r$ and the union of all $Y \subseteq H$ which give rigid extensions of $x_1, \ldots, x_r$. In Figure 6 $H^- = \{x_1, x_2, z_1\}$. By assumption $y \notin H^-$. But then (H^-, H) is safe: if there were H^{-+}, $H^- \subset H^{-+} \subseteq H$ with (H^-, H^{-+}) dense then there would be H^{-+-} with $H^- \subset H^{-+-} \subseteq H^{-+}$ and (H^-, H^{-+-}) rigid (recall, every dense extension has a rigid subextension) and then (R, H^{-+-}) would be rigid (the rigid extension of a rigid extension being rigid), contradicting the definition of H^- as the union of all such rigid extensions.

Now we claim

$$\vdash_T (\exists y)\mathrm{cl}_t(x_1, \ldots, x_r, y) \cong H \Leftrightarrow \mathrm{cl}_t(x_1, \ldots, x_r) \cong H^-$$

As the s-closure determines the t-closure this will imply the Key Lemma. The implication $\Rightarrow$ is immediate as the t-closure of $x_1, \ldots, x_r, y$ determines

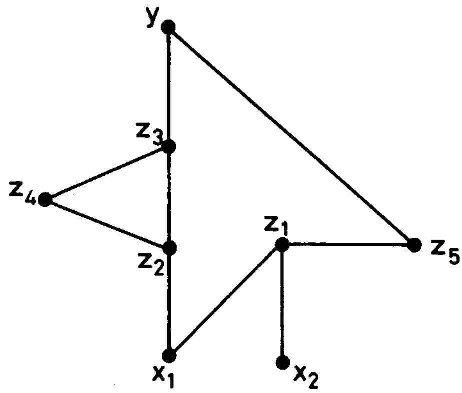


Figure 6. $H = \text{cl}_3(x_1, x_2, y)$ *outside*

the t-closure of $x_1, \ldots, x_r$. The implication the other way makes full use of the Generic Extension Schema. As (H^-, H) is safe there exist $y, z_1, \ldots, z_p$ giving an (H^-, H) extension over H^- with precisely the edges of H and with $\text{cl}_t(x_1, \ldots, x_r, y, z_1, \ldots z_p) = \text{cl}_t(x_1, \ldots, x_r) \cup \{y, z_1, \ldots, z_p\} = H$. Thus $\text{cl}_t(x_1, \ldots, x_r, y) \subset H$ but H, as a potential value of $\text{cl}_t(x_1, \ldots, x_r, y)$ has all z's lying in rigid extensions of size at most t over $x_1, \ldots, x_r, y$ and hence $\text{cl}_t(x_1, \ldots, x_r, y) = H$.

Completeness Theorem. T is complete.

We prove by induction on length that for any formula $P(x_1, \ldots, x_r)$ there exists $t = t(P)$ so that for any possible value H of $\text{cl}_t(x_1, \ldots, x_r)$ either

$$\vdash_T \text{cl}_t(x_1, \ldots, x_r) \cong H \Rightarrow P(x_1, \ldots, x_r)$$

or

$$\vdash_T \text{cl}_t(x_1, \ldots, x_r) \cong H \Leftarrow \neg P(x_1, \ldots, x_r)$$

Formally, formulae may be build up from atomic formulas using only $\neg, \vee$ and $\exists y$. For the atomic formulae $x_i = x_j$ and $x_i \sim x_j$ we take $t = 0$. For $Q = \neg P$ we take $t(Q) = t(P)$. For $Q = P_1 \vee P_2$ we take $t(Q) = \max[t(P_1), t(P_2)]$. The essential case is when Q is of the form $(\exists y)P(x_1, \ldots, x_r, y)$. Here $t(Q) = s$ is given by the Key Lemma.

Now let P be a sentence — i.e., a formula with no free variables. There exists t so that for any possible value H of $\mathrm{cl}_t(\emptyset)$ either

$$\vdash_T \mathrm{cl}_t(\emptyset) \cong H \Rightarrow P \quad \text{or} \quad \vdash_T \mathrm{cl}_t(\emptyset) \cong H \Rightarrow \neg P$$

But the Nonexistence Schema, in the form of the Null Closure Schema, Gives $\mathrm{cl}_t(\emptyset) = \emptyset$ as a theorem of T. Hence either P or $\neg P$ is a theorem of T.

3. The model

Here we give a countable model G for T. In particular, this implies the consistency of T.

Remark. Assume, as shown in [1], that all sentences A of T holds a.s. in $G(n,p)$ with $p = n^{-\alpha}$. For any finite set $A_1, \ldots, A_r$ of such sentences let n be so large (unravelling the limit definition) that

$$\Pr[G(n,p) \models A_i] > 1 - r^{-1}, \quad 1 \le i \le r$$

For this n $\Pr[G(n,p) \models A_1 \wedge \ldots \wedge A_r] > 1 - r/r = 0$ so there exists a specific graph G on n vertices satisfying $A_1, \ldots, A_r$. As every finite segment of T has a model, T is consistent by a standard compactness argument.

The vertex set of G will be the set N of positive integers. Make a countable list $\mathcal{Z}$ of all pairs $((R,H),A)$ consisting of a safe rooted graph (R,H) and a set $A \subset N$ with $|A| = |R|$. We build G in stages $G^0, G^1, G^2, \ldots$, with G^0 being the null graph on the null set. Suppose that G^i is defined on $V^i = 1, 2, \ldots, n_i$. Let $((R,H),A)$ be the next pair on the list $\mathcal{Z}$. If $A \not\subseteq V^i$ reorder the list so that it is. (In the list there are a countable number of entries with $R = A = \emptyset$ that may be used.) Say H has vertices $a_1, \ldots, a_r, b_1, \ldots, b_v$ of which the a's are in R. To G^i add the vertices $n_i + 1, \ldots, n_i + s$ and make them an (R,H) extension of A. That is, with $A = \{x_1, \ldots, x_r\}$ make x_j adjacent to $n_i + k$ when a_j is adjacent to b_k in H and make $n_i + j$ adjacent to $n_i + k$ when b_j is adjacent to b_k in H. Add no other edges. This defines G^{i+1} with $n_{i+1} = n_i + s$.

Observe that once G^i is defined no further edges on V^i are created so that this process creates an infinite graph G on N. We claim G is a model for T.

Let $x_1, \ldots, x_r \in V^i$. We claim $\mathrm{cl}_t(x_1, \ldots, x_r) \subset V^i$. Suppose not. There is a rigid extension by some $y_1 < \ldots < y_s$ with $y_s \notin V^i$. For some j, p then $y_1, \ldots, y_p \in V^{j-1}$ and $y_{p+1}, \ldots, y_s \in V^j - V^{j-1}$. Then $y_{p+1}, \ldots, y_s$ in a rigid extension of $x_1, \ldots, x_r, y_1, \ldots, y_p$. Adding roots does not affect rigidity so $y_{p+1}, \ldots, y_s$ is rigid over V^{j-1}. But (V^{j-1}, V^j) is safe since the only edges added gave a safe (R, H) extension. Hence any $Y \subset V^j - V^{j-1}$ gives a safe extension over V^{j-1} and safe cannot be rigid — a contradiction. Thus the t-closure as defined in G^i is the t-closure as defined in G.

In G^0, $\mathrm{cl}_t(\emptyset) = \emptyset$. Thus in G $\mathrm{cl}_t(\emptyset) = \emptyset$ so the Nonexistence Schema, in the form of the Null Closure Schema, is satisfied. Let $x_1, \ldots, x_r \in N$ and let (R, H) be safe with $|R| = r$. Let i be minimal with $x_1, \ldots, x_r \in V^i$. In the construction of G for some $j > i$ $V^j - V^{j-1} = \{y_1, \ldots, y_v\}$ makes an (R, H) extension of $x_1, \ldots, x_r$. We claim this extension is t-generic for all t. We may consider t-closures as defined in V^j. Consider a rigid extension $z_1, \ldots, z_p$ over $x_1, \ldots, x_r, y_1, \ldots, y_v$ in V^j. The y's are adjacent only to x's and other y's, hence not to the z's. Hence the z's must from a rigid extension over the x's. Thus $\mathrm{cl}_t(x_1, \ldots, x_r, y_1, \ldots, y_v) = \mathrm{cl}_t(x_1, \ldots, x_r) \cup \{y_1, \ldots, y_v\}$ as claimed. Thus G satisfies the Generic Extension Schema.

Remark. In [3], where this result is given in more detail, it is also shown that there are uncountable many countable models for T.

References

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