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The Graph Algebra of Skew Hadamard Determinants

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Suppose all entries in an $n \times n$ determinant satisfy $|a_{ij}| \leq 1$; what is the maximal value of $D_n = |\det a_{ij}|$? From Hadamard's classical determinant inequality it follows that $D_n \leq n^{\frac{1}{2}n}$, equality occurring if and only if $|a_{ij}| = 1$ for all i, j, and all rows (or columns) are orthogonal to each other. This prompted Hadamard [1] in 1893 to ask for the values of n for which such determinants exist.

Suppose D_n is real; then all entries must be ± 1 and Hadamard's problem is equivalent to Sylvester's earlier question ([3], 1867) whether there exists a ± 1 "inverse orthogonal" matrix for all orders $n \equiv 0 (\text{mod } 4)$, that is one in which all rows (or columns) are orthogonal to each other. This is possibly the oldest unsolved combinatorial problem in existence. It is easy to see that for $n > 2$ the condition $n \equiv 0 (\text{mod } 4)$ is necessary for the existence of Sylvester's matrices, now traditionally named after Hadamard.

Since then almost all work on Sylvester's problem has been concentrated on the explicit construction of inverse orthogonal ± 1 matrices of various orders, and related combinatorial designs. Only a small fraction of the work has been concerned directly with Hadamard's original problem. In 1935 Turán suggested to me that perhaps it would be possible to calculate the sum of the $2N$-th powers of all ± 1 determinants of order n; if this were possible, then taking the $2N$-th root of the result would converge (as

N $\to\infty$) directly to the maximal value all ±1 determinants of n-th order. We were indeed able to evaluate the sum when $N=1$ and 2 for all n [4].

Much later I noted [5] that Turán's method might become more manageable if we restrict it to skew symmetric determinants. So let us ask: What is the maximal value of a real skew symmetric determinant of order $2k$, with all entries $|a_{ij}|\le 1$. The diagonal entries are of course 0, and so is the determinant of a skew matrix of odd order. Hadamard's inequality now tells us that $D_{2k}\le(2k-1)^k$, equality if and only if all off-diagonal entries are ±1 and the matrix of the determinant is inverse orthogonal. Since a skew determinant is the square of a certain form of the entries with integer coefficients, $D_{2k}=(2k-1)^k$ for $k>1$ can only happen if k is even, but we can ask generally for the maximum D_{2k} for all k, even if we cannot suggest an expected value for D_{2k} when k is odd. A proof of $D_{4k}=(4k-1)^{2k}$ would immediately imply the existence of a skew type Hadamard matrix of order $4k$.

Let (a_{ij}) be a skew symmetric ±1 matrix, that is $a_{ij}=-a_{ij}$ for all i,j and $a_{ij}^2=1$ for $i\ne j$. Let $T_{2k}\subset S_{2k}$ be the set of permutations on $K=\{1,2,\dots,2k\}$ which are products of disjoint cycles of even lengths. Then (see e. g. [2], chapter 9)

$$\det\,(a_{ij})=\sum_{\pi\in T_{2k}}(-1)^{\nu(\pi)}a_\pi$$

where $\nu(\pi)$ is the number of disjoint cycles of π and a_π is defined as follows: For a cycle $\gamma=(k_1k_2\dots k_{2i})$, $a_\gamma=a_{k_1k_2}a_{k_2k_3}\dots a_{k_{2i}k_1}$ and if π is the product of disjoint cycles $\gamma_1,\gamma_2,\dots,\gamma_p$ then $a_\pi=a_{\gamma_1}a_{\gamma_2}\dots a_{\gamma_p}$. But $a_{ij}a_{ji}=-1$ hence the general term in the determinant is $(-1)^{\mu(\pi^*)}a_{\pi^*}$ where π^* is the (possibly empty) product of disjoint even cycles of length >2 and $\mu(\pi^*)$ is the number of these cycles. Let $\rho=\rho(\pi)$ be the number of 2-cycles in π so that $\rho(\pi)+\mu(\pi)=\nu(\pi)$, then for fixed π^* there are $1\times3\times\dots\times(2\rho-1)=\dfrac{(2\rho)!}{2^\rho\rho!}=P(\rho)$ permutations $\pi\in T_{2k}$ with this π^*. We also note that if $\overline{\gamma}=(k_{2i}k_{2i-1}\dots k_2k_1)$ is the inverse of γ then $a_{\overline{\gamma}}=a_\gamma$ since $a_{ji}=-a_{ij}$ and there are an even number of such inversions. Therefore we don't have to list γ and $\overline{\gamma}$ separately, and with every $\gamma_1\gamma_2\dots\gamma_\nu$ we have 2^μ equivalent ones giving the same $a_{\gamma_1\dots\gamma_\mu}$.

Let T_{2k}^* be the set of essentially different products $\pi^*=\gamma_1\gamma_2\dots\gamma_\mu$ of total length $|\pi^*|=2(k-\rho)$, $0\le\rho\le k$ (the empty product included) then the determinant is

$$\det (a_{ij}) = \sum_{\pi^* \in T^*_{2k}} (-2)^{\mu(\pi^*)} P(\rho(\pi^*)) a_{\pi^*}.$$

Take for instance $k = 2$. T_4 consists of

$$(12)(34), (13)(24), (14)(23), (1234), (1432), (1342), (1243), (1423), (1324)$$

hence

$$\begin{aligned} \det =& a_{12}a_{21}a_{34}a_{43} + a_{13}a_{31}a_{24}a_{42} + a_{14}a_{41}a_{23}a_{32} \\ & - (a_{12}a_{23}a_{34}a_{41} + a_{14}a_{43}a_{32}a_{21} + a_{13}a_{34}a_{42}a_{21} + \\ & a_{12}a_{24}a_{43}a_{31} + a_{14}a_{42}a_{23}a_{31} + a_{13}a_{32}a_{24}a_{41}) \\ =& 3 - 2\Sigma \end{aligned}$$

where $\Sigma = a_{(1234)} + a_{(1342)} + a_{(1423)}$.

Let us calculate $(\det)^N = (3 - 2\Sigma)^N$. We have to know Σ^2. Now

$a_{(1234)}a_{(1234)} = 1$, $a_{(1234)}a_{(1342)} = -a_{(1423)}$, $a_{(1234)}a_{(1423)} = -a_{(1342)}$ as seen e.g. from

$$a_{(1234)}a_{(1342)} = a_{12}a_{23}a_{34}a_{41}a_{13}a_{34}a_{42}a_{21} = -a_{23}a_{41}a_{13}a_{42} = -a_{(1423)}.$$

Replacing $a_{(1234)}$ by $a_{(1342)}$ and $a_{(1423)}$ respectively, and adding, we find $\Sigma^2 = 3 - 2\Sigma$.

Set $(\det)^N = (3 - 2\Sigma)^N = b_N - c_N\Sigma$. We get

$$\begin{aligned} b_{N+1} - c_{N+1}\Sigma &= (b_N - c_N\Sigma)(3 - 2\Sigma) \\ &= 3b_N - 3c_N\Sigma - 2b_N\Sigma + 2c_N(3 - 2\Sigma), \\ b_{N+1} = 3b_N + 6c_N, &\qquad c_{N+1} = 2b_N + 7c_N, \end{aligned}$$

solved by $b_N = \frac{1}{4}(9^N + 3)$, $c_N = \frac{1}{4}(9^N - 1)$. This shows that $\lim_{N\to\infty} (b_N - c_N\Sigma)^{1/N} = 9$ which is therefore the maximal value of $3 - 2\Sigma$, as it should be (since $9 = (4-1)^2$). Note that to get this result it was not necessary to actually solve for b_N, c_N. If B is the matrix $B = \begin{pmatrix} 3 & 6 \\ 2 & 7 \end{pmatrix}$ then $\begin{pmatrix} b_N \\ c_N \end{pmatrix} = B^{N-1} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and all that matters is the maximal eigenvalue of B, namely 9.

In the general case we want to calculate

$$\left(\sum_{\pi^* \in T^*_{2k}} (-2)^{\mu(\pi^*)} P(\rho(\pi^*)) a_{\pi^*}\right)^N .$$

The numerical factor in front of a_{π^*} depends only on the cycle structure of π^* and the factor multiplies $\sum_{\pi^*} a_{\pi^*}$, summed for all π^* with the same cycle structure, and characterized by the lengths of the cycles, $(2m_1, 2m_2, \ldots, 2m_\mu)$ where

$$2 \leq m_1 \leq m_2 \leq \ldots m_\mu, \qquad m_1 + m_2 + \ldots + m_\mu \leq k.$$

Let Π_k be the set of these partitions $\mathbf{m}$, $M_{\mathbf{m}}$ the set of $\pi^* \in T^*_{2k}$ with cycle structure $\mathbf{m}$, $Q_{\mathbf{m}}$ the corresponding factor $Q_{\mathbf{m}} = (-2)^\mu P(\rho) = (-2)^\mu \dfrac{(2\rho)!}{2^\rho \rho!}$ where $\rho = k - (m_1 + \ldots + m_\mu)$. Then we want

$$\left(\sum_{\mathbf{m} \in \Pi_k} Q_{\mathbf{m}} \sum_{\pi^* \in M_{\mathbf{m}}} a_{\pi^*}\right)^N .$$

To compute this expression we need a multiplication table for the $\sum_{\pi^* \in M_{\mathbf{m}}} a_{\pi^*}$. Now the product of two a_{π^*} is a (possibly empty) signed product of the a_{ij} in which no a_{ij} appears more than once. Such a term can be represented by an oriented graph in which every vertex has an even total degree. The algebra of the expressions $\sum_{\pi^* \in M_{\mathbf{m}}} a_{\pi^*}$ can be replaced in an obvious manner by an algebra of graphs with a suitable rule of multiplication which we are now going to describe.

A (non-directed) graph Γ on $2k$ vertices will be called an E-graph if it has no loops or double edges, and the edge valency of each vertex is even (possibly 0). In particular the empty graph (with $2k$ vertices but no edges) is an E-graph. A connected E-graph is of course Eulerian.

Given a non-empty E-graph Γ with e edges we can equip the edges with an orientation in 2^e ways. These orientations fall in two classes according to the parity of the number of reversals needed to carry one orientation into the other. We call such a class of oriented graphs a *signed* E-graph, with underlying graph Γ. If γ is a signed graph then its companion class, called its conjugate, will be denoted $-\gamma$.

Suppose γ is a signed E-graph on $K = \{1, 2, \ldots, 2k\}$. An automorphism of γ is a permutation of its labels which carries γ into itself or into $-\gamma$. If every automorphism carries γ into itself then γ is called *proper*. This is certainly the case if γ has no non-trivial automorphism. Otherwise (if it is not proper) exactly half of the automorphisms carry γ into $-\gamma$. For instance the signed graph on $\{1,2,3,4,5,6\}$ represented by the directed edges

$$1 \to 2, \quad 1 \to 3, \quad 1 \to 4, \quad 1 \to 5, \quad 2 \to 6, \quad 3 \to 6, \quad 4 \to 6, \quad 5 \to 6$$

has 48 automorphisms generated by the transposition $(1,6)$ and permutations of the set $\{2,3,4,5\}$. It is easily seen that they all carry γ into itself and so γ is proper. On the other hand the triangle $1 \to 2$, $2 \to 3$, $3 \to 1$ (and indeed any circuit of odd lengths) is carried into its conjugate by an odd permutation of the vertices, hence is not proper. E-graphs which are unions of disjoint circuits of even lengths are proper.

Given k let $G = G_k$ be the set of all proper signed E-graphs on K, including the empty graph γ_0. $H = H_k$ denotes the subset of those members of G which are disjoint unions of circuits of even lengths. We define a multiplication on G as follows: the product $\gamma_1\gamma_2$ is the "signed symmetric difference" of γ_1 and γ_2, that is the union $\gamma_1 \cup \gamma_2$ from which the common edges and circuits of length 2 (both illegal) have been removed. The remaining edges form a signed graph γ_3 and if the number of removed 2-circuits was r (we do not count the removed common edges) then $\gamma_1\gamma_2 = (-1)^r\gamma_3$. In particular $\gamma_0\gamma = \gamma\,\gamma_0 = \gamma$ for all $\gamma \in G$. This multiplication is clearly associative and commutative and we can form the (associative and commutative) algebra $\mathbb{Z}G$ with unit γ_0 and the condition $(-1)\gamma = -\gamma$.

We consider now the subalgebra A_k generated by the elements

$$\sigma_\gamma = \frac{1}{|\mathrm{aut}(\gamma)|} \sum_{\pi \in S_{ik}} \gamma^\pi \tag{1}$$

where $|\mathrm{aut}(\gamma)|$ is the number of automorphisms of γ and γ^π is the graph obtained from γ by applying the permutation π to its vertices. Note that for a non-proper γ the sum in (1) would have been 0. On the other hand for a proper γ each γ^π appears $|\mathrm{aut}(\gamma)|$ times in the sum, therefore $\sigma_\gamma \in \mathbb{Z}G$. The unit of A_k is $\sigma_{\gamma_0} = 1$.

If we disregard orientation and labelling, all members in the sum (1) are isomorphic. Let $\mathbf{G}$ be the set of non-directed, unlabeled proper E-graphs on $2k$ vertices, $\Phi : G \to \mathbf{G}$ the natural embedding of G into $\mathbf{G}$. If $\Gamma = \Phi(\gamma)$, we can write σ_i for σ_Γ. Then the elements σ_γ, $\Gamma \in \mathbf{G}$ from a basis for A_k, with multiplication table

$$\sigma_{\Gamma_i}\sigma_{\Gamma_j} = \sum_{p=0}^{n} C_{ijp}\sigma_{\gamma_p}, \qquad C_{ijp} = C_{jip} \in \mathrm{Z} \tag{2}$$

where $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$ are the elements of $\mathbf{G}_k$. Once their order is fixed we can just as well write σ_i for σ_{Γ_i}, $\sigma_0 = 1$. Concerning the ordering of the Γ_i we make the convention that we first list those which are in $\mathbf{H}_k$, the set of (non-oriented, unlabelled) graphs which are disjoint unions of even circuits of length > 2. These are characterized by partitions $\mathbf{m} \in \Pi_k$, the half-lengths of the disjoint circuits appearing in the graph. The link between maximal skew determinant and the algebra A_k is now obvious and is expressed by the following theorem.

Theorem. *Let the matrix $B = B_k = (B_{ij})$ be defined*

$$B_{jp} = \sum_{i=0}^{m} Q_i C_{ijp} \tag{3}$$

where m is the number of distinct partitions $\mathbf{m} \in \Pi_k$, $Q_i = Q_{\mathbf{m}_i} = (-2)^{\mu}\dfrac{(2\rho)!}{2^{\rho}\rho!}$ where $\rho = k - (m_1 + \ldots + m_\mu)$, and C_{ijp} are the structure coefficients in (2). Then the maximal eigenvalue of B_k is equal to the maximal ± 1 skew determinant of order $2k$.

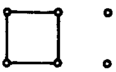
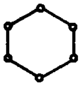
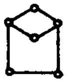
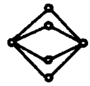
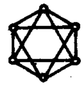
The theorem reduces the existence problem of skew Hadamard matrices to the computation of the structure constants of A_k. Indeed not all coefficients are needed, only those C_{ijp} which refer to products of the members of $\mathbf{H}$ with members of $\mathbf{G}$. For illustration we take the cases $k = 2$ and 3.

(i) $k = 2$. This has already been treated earlier but we repeat the calculation in the graph setting. $\mathbf{G}$ (and $\mathbf{H}$) has only one non-trivial member, namely the circuit of length 4. Hence $m = n = 1$, and Γ_1 is the graph □. It has 8 automorphisms and σ_1 has 24/8=3 terms, namely (using the cycle notation) $\gamma = 1 \to 2 \to 3 \to 4 \to 1 = (1234)$, $\gamma' = (1342)$, $\gamma'' = (1423)$. Let us calculate the only non-trivial product, namely σ_1^2. Multiply γ with the three terms of σ_1. We get γ_0 from (1234)(1234), $-\gamma''$ from (1234)(1342) and $-\gamma'$ from (1234)(1423). Replacing (1234) by (1342), then (1423), and adding all the terms we obtain $\sigma_1^2 = 3\sigma_0 - 2\sigma_1$, as before. This gives $C_{000} = 1$, $C_{101} = C_{011} = 1$, $C_{110} = 3$, $C_{111} = -2$, all other structure coefficients are 0. Also $\mu_0 = 0$, $Q_0 = 3$, $\mu_1 = 1$, $Q_1 = -2$, and we get

$$B_2 = \begin{pmatrix} 3 & -2 \\ -6 & 7 \end{pmatrix}$$

with eigenvalues 1 and 9. It follows from the theorem that the maximal ± 1 skew determinant of order 4 is 9, as it should be.

(ii) $k = 3$. **G** has five non-trivial members, namely

Γ_1 :  Γ_2 :  Γ_3 :  Γ_4 :  Γ_5 : 

the first two being members of **H**.

By a calculation similar to the one before we obtain the following multiplication table:

$$\begin{aligned}
\sigma_1\sigma_1 &= 45\sigma_0 + 6\sigma_1 - 6\sigma_2 + 2\sigma_3 + 6\sigma_4 \\
\sigma_2\sigma_1 &= \sigma_1\sigma_2 = -8\sigma_1 - 3\sigma_2 - 2\sigma_3 \\
\sigma_1\sigma_3 &= 8\sigma_1 - 6\sigma_2 - 3\sigma_3 + 12\sigma_4 + 12\sigma_5 \\
\sigma_1\sigma_4 &= 2\sigma_1 + \sigma_3 + 3\sigma_5 \\
\sigma_1\sigma_5 &= \sigma_3 + 3\sigma_4 - 6\sigma_5 \\
\sigma_2\sigma_2 &= 60\sigma_0 - 4\sigma_1 - 8\sigma_2 - 4\sigma_3 + 12\sigma_4 + 12\sigma_5 \\
\sigma_2\sigma_3 &= -8\sigma_1 - 12\sigma_2 - 8\sigma_3 + 24\sigma_5 \\
\sigma_2\sigma_4 &= 3\sigma_2 \\
\sigma_2\sigma_5 &= 3\sigma_2 + 2\sigma_3 - 8\sigma_5.
\end{aligned}$$

This is not the complete multiplication table but it suffices since the structure coefficients C_{ijp} are only needed for $i = 0, 1$ and 2. Noting $\mu_0 = 0$, $\mu_1 = \mu_2 = 1$, $Q_0 = 15$, $Q_1 = Q_2 = -2$, we obtain

$$B_3 = \begin{pmatrix} 15 & -2 & -2 & 0 & 0 & 0 \\ -90 & 19 & 18 & 0 & -12 & 0 \\ -120 & 24 & 37 & 12 & -24 & -24 \\ 0 & 0 & 36 & 37 & -24 & -72 \\ 0 & -4 & -6 & -2 & 15 & -6 \\ 0 & 0 & -6 & -6 & -6 & 43 \end{pmatrix}$$

The matrix has eigenvalues 1, 1, 9, 25, 49, 81 (that is all the odd squares from 1 to 9^2, with 1 repeated). It follows from the theorem that the maximal ± 1 skew determinant of order 6 is 81. This is lower than the Hadamard bound $5^3 = 125$, as it should be since there are no orthogonal ± 1 skew determinants of order 6. The other eigenvalues are just the possible values of ± 1 skew determinants of order 6 (they are necessarily squares). The multiplicity of the eigenvalues appearing in the matrix is presumably the number of non-equivalent skew symmetric matrices with that determinant.

Note added in proof. Recent computer examination of the case $k = 4$ has shown that there are 69 proper E-graphs on 8 vertices (including the empty graph) and the B-matrix obtained from their multiplication table has the eigenvalues $(2k-1)^2$, $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 25$, with corresponding multiplicities 13, 9, 6, 8, 7, 4, 3, 4, 4, 2, 2, 1, 1, 1, 1, 1, 1, 1. It can be easily verified that these are exactly the values of the skew ± 1 determinants of order 8. The multiplicities seem to be related to the number of non-equivalent skew ± 1 matrices with the respective determinants (equivalence under multiplication by -1 of rows and corresponding columns, and simultaneous permutation of rows and corresponding columns).

References

[1] J. Hadamard, Resolution d'üne question relative aux determinants, *Bull. des Sciences Mathématiques* **17**(1893), 240–246.

[2] G. Kowalewski, *Einführung in die Determinantentheorie*, Chelsea, New York, 1948.

[3] J. J. Sylvester, Thoughts on inverse orthogonal matrices, *Phil. Mag.* (4) **24**(1867), 461–475.

[4] Gy. Szekeres, and P. Turán, Egy szélsőérték feladat a determinánselméletben, *Mat. Természettud. Értesítő* **16**(1937), 796–806.

[5] G. Szekeres, Determinants of skew type ± 1 Matrices, *Periodica Mat. Hungarica* **3**(1973), 229–234.

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