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Combinatorial Properties of Certain Classes of 3-polytopal Planar Graphs

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1. Introduction

Consider a planar graph G. An edge h of G is of type $(a, b; m, n)$ if its vertices are of degrees a and b and the two faces incident with h are an m-gon and an n-gon. The present paper deals with planar 3-connected (i.e. 3-polytopal) graphs having edges of exactly two types. Let $S(a, b, c; m, n, k)$ denote the class of 3-polytopal graphs with edges of types $(a, b; m, n)$ and $(b, c; n, k)$. In [4] the first step in the study of the combinatorial structure of such graphs has been made. The questions of the existence of such graphs were solved. The cardinalities of all classes were determined. In the present paper we continue the investigation of the combinatorial structure of graphs from some classes of graphs with two types of edges.

For any graph G, let $v_i(G)$ or v_i denote the number of i-valent vertices of G and $s_i(G)$ or s_i denote the number of i-gons of G, then $(v_i(G))$ and $s_i(G)$ is the vertex-vector and the face-vector of G, respectively. (In the sequel the superfluous zeros will be left out.)

Let R denote the class of 3-polytopal graphs with edges of types $(4, 4; 3, 5)$ and $(4, 6; 5, 4)$ and let T denote the class of graphs dual to those from R. It is easy to see, that $R = S(4, 4, 6; 3, 5, 4)$ and $T = S(3, 5, 4; 4, 4, 6)$. In [4] it was shown, that both of these classes contain infinitely many graphs.

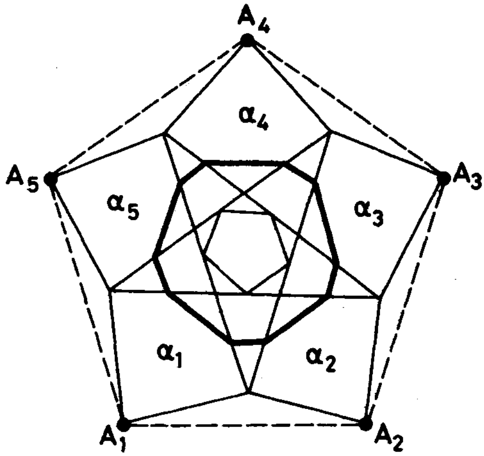

Figure 1.

In this paper we characterize all the graphs from R (or similarly from T): they can be generated from the medial (or from the radial) graph of the dodecahedron (see [6], [5] or Fig. 1) using one operation.

More precisely, let G_0 denote the medial graph of the dodecahedron, shown in Fig. 1. It is a planar 3-connected graph with all edges of type $(4,4;3,5)$ and it is easy to see that it is the unique graph with these properties. Let L be the configuration obtained from G_0 by deleting all "dashed" edges in Fig. 1.

Let G_1 denote the graph from the class R obtained as follows: Embed in every face of a dodecahedron graph a configuration L in such a way that the vertices $A_1, A_2, \ldots, A_5$ coincide with the vertices of the face. Then delete the original edges of the dodecahedron graph.

Let K denote the configuration marked by a thick line boundary in Fig. 1 and let N denote the configuration obtained from G_1 by deleting one (any one) copy of configuration K.

We shall say that a planar graph G has been "enlarged" by the operation $\mathbb{R}$, when the configuration K has been substituted with the configuration N. We shall say that a planar graph has been "reduced" by the operation $\mathbb{R}^{-1}$ when the configuration N has been substituted with the configuration K. The main results are summed in the following theorem.

Theorem. *Let G be a graph from R. Then there is the integer $n \geq 1$ such that:*

1. *There is a sequence of graphs $G_0, G_1, \ldots, G_n$, such that G_0 is the medial graph of the dodecahedron, $G_1, \ldots, G_n$ are from R, $G_n = G$, and each G_{i+1} can be obtained from G_i by the operation $\mathbb{R}$.*
2. *The vertex-vector (v_4, v_6) of G and the face-vector (s_3, s_4, s_5) of G, satisfy the following conditions:*

$$v_4 = 30 + 270n, \qquad v_6 = 20n, \qquad s_3 = 20 + 160n, \quad s_4 = 30n, \quad s_5 = 12 + 120n.$$

Remark. It is easy to see that, by using the duality, we can obtain similar results for all the graphs from the class T.

2. Proof of Theorem

We begin by presenting three useful lemmas.

Lemma 1. *There exist no 3-valent connected planar graph with the property that all but one of its faces are pentagons, while one "exceptional" face is not the pentagon.*

Proof. Let G (if possible) be 3-valent connected planar graph which contains exactly one m-gon ($m \neq 5$) besides pentagons. B. Grünbaum in [2, Theorem 2(k), 3(k), 4(k), p. 272] showed, that:
If such graph G exists, then

(i) $m \equiv 0 \pmod 5$)

(ii) G is 2-connected,

(iii) $s(G) \equiv 2 \pmod{10}$.

From the famous Euler's formula

$$\sum_{i\geq 2}(6-i)s_i + 2\sum_{j\geq 2}(3-j)v_j = 12$$

it follows that

$$s_5(G) = 6 + m. \tag{1}$$

From (ii) and (iii) it follows, that G cannot contain any edge of type $(3,3;m,m)$ and that (by using (1)) $7 + m \equiv 2 \pmod{10}$ and $m \geq 15$, respectively. Now we show that the assumption: G contains the pentagon with two edges of type $(3,3;5,m)$, leads to a contradiction. Let α (if possible) be such face of G. Let e and f denote two edges of type $(3,3;5,m)$ from α and let x denote the vertex from α which is not incident either with e nor with f. Two vertices from α which are adjacent with x we denote y and z. Let G_1 be the graph obtained from the connected component of the graph $G - \{e, f\}$, which contains the vertex x, by adding the edges yz.

It is clear that G_1 is connected, 3-valent, planar graph, all faces of which are pentagons except one (or two adjacent) exceptional face(s) and one exceptional face is a triangle. In [2, Theorem 2(k)] it was shown, that no such graph G_1 can exist.

Each pentagon from G contains at most one edge of type $(3,3;5,m)$ and so G must contain exactly m pentagons which are adjacent with m-gon. From (1) it follows that G contains exactly six pentagons which are not incident with m-gon. It is easy to see that it is not possible if $m \geq 15$. The lemma follows. ■

Corollary 1. *For every $m \neq 5$ there exists no graph which contains exactly one m-gon in the class $S(4,4,4;5,3,m)$.*

Proof. Let G (if possible) be the graph from $S(4,4,4;5,3,m)$ which contains exactly one m-gon ($m \neq 5$). Associate the graph $L(G)$ to the graph G as follows: The vertices of $M(G)$ will be the vertices associated with the triangular faces of G. Any edge of $L(G)$ will join two vertices if triangular faces corresponding to them have a common vertex in G.

Evidently, $L(G)$ is 3-valent connected planar graph which contains exactly one m-gon ($m \neq 5$) besides pentagons. From Lemma 1 it follows that such a graph G cannot exist. ■

Lemma 2. *If G is a graph from R, then G has at least one copy of configuration N.*

Proof. Let G be the graph from R. Associate the graph $M(G)$ to the graph G as follows: The vertices of $M(G)$ will be the vertices associated with the 6-valent vertices of G. Any edge of $M(G)$ will join two vertices if vertices corresponding to them are incident with the same quadrangular face in G.

Let μ be a connected component of $M(G)$ none face of which, with the exception of the "periphery", contains another component of $M(G)$.

Evidently, μ contains no 2-gons, since this would necessitate the graph G to be 2-connected. Now we show that μ does not contain any m-gon ($m \neq 5$) as its interior face. (By the interior faces of μ we mean all faces of μ except the periphery.)

Let γ be the m-gonal interior face of μ (if possible). We denote by $A'_1, A'_2, \ldots, A'_m$ the vertices of γ and by $A_1, A_2, \ldots, A_m$ the corresponding vertices to them in G (see Fig. 2a). Let G_1 be the graph obtained from G by adding m "new" edges (the edge A_iA_{i+1} for $i =, \ldots, m-1$ and the edge A_1A_m). The graph G_1 contains, as its subgraph, the configuration (we denote it K_m) marked by a "dashed" line boundary in Fig. 2b. All vertices of K_m are 4-valent and all faces of K_m are either triangles or pentagons (faces $\alpha_1, \alpha_2, \ldots, \alpha_m$ are necessarily pentagons). It is easy to see, that K_m is 3-connected graph from $S(4,4,4;5,3,m)$, which contains exactly one m-gon. Using Corollary 1 we see that no such a graph can exist. Therefore all the interior faces of μ are pentagons.

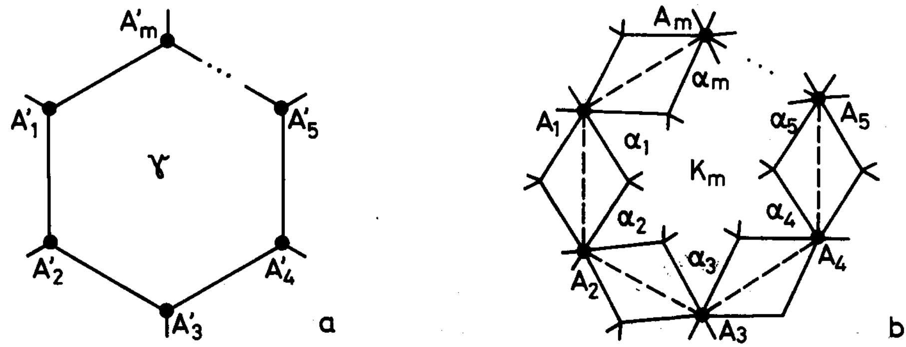

Figure 2.

Let k denote the number of sides of the periphery of μ. The assumption $k \neq 5$ leads in a contradiction to the Lemma 1. This implies that μ is the graph of a dodecahedron. The existence of pentagons in μ (excepting possibly the periphery of μ) means that in the graph G there is a configuration L (see Fig. 1) with $A_1, A_2, \ldots, A_5$ 6-valent vertices and $\alpha_1, \alpha_2, \ldots, \alpha_5$ pentagons. It is easy to verify that the existence of μ in $M(G)$ means that in the graph G there is a copy of the configuration N. The Lemma follows. ∎

There is an easy corollary of the before said lemma.

Corollary 2. *If G is a graph from R, then G has at least eleven copies of the configuration L.*

It is not difficult to prove the following lemma.

Lemma 3. *If G is a graph from R, then making in G one of the operations $\mathbb{R}$ or $\mathbb{R}^{-1}$ we get again a graph from R, except the possibility that $G = G_1$ and we use the operation $\mathbb{R}^{-1}$; in this case we get the graph G_0.*

Now we consider the graph G from R and denote by G^{-1} the graph obtained from G by using the operation $\mathbb{R}^{-1}$. An easy calculation shows that $v_4(G^{-1}) = v_4(G) - 270$, $v_6(G^{-1}) = v_6(G) - 20$, $s_3(G^{-1}) = s_3(G) - 160$; $s_4(G^{-1}) = s_4(G) - 30$; $s_5(G^{-1}) = s_5(G) - 120$.

If G^{-1} is from R, then (by using Lemma 1) we can make again in G^{-1} the operation $\mathbb{R}^{-1}$. If G^{-1} is not from R, then (by using Lemma 2) $G^{-1} = G_0$. Observe that $v_4(G_0) = 30$, $s_3(G_0) = 20$ and $s_5(G_0) = 12$.

From the before said the theorem follows immediately. ■

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