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Decompositions into Subhypergraphs with the König Property

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ABSTRACT

Given a hypergraph $\mathbf{H}$, we provide upper bounds on the smallest number, t, of hypergraphs $\mathbf{H}_i$ with $\mathbf{H}_1 \cup \ldots \cup \mathbf{H}_t = \mathbf{H}$ such that in each $\mathbf{H}_i$ the maximum number, $\nu(\mathbf{H}_i)$, of pairwise disjoint edges is equal to the minimum number, $\tau(\mathbf{H}_t)$ of vertices meeting all edges of $\mathbf{H}_i$. One of our conjectures states that if $\mathbf{H}_i$ is k-uniform, $k \geq 3$, and $\tau(\mathbf{H}') = \nu(\mathbf{H}')$ holds for every $\mathbf{H}' \subset \mathbf{H}_i$, $1 \leq i \leq t$, then $\mathbf{H}_1 \cup \ldots \cup \mathbf{H}_t$ cannot contain a complete hypergraph on $k+t$ vertices.

1. Introduction

A *hypergraph* $\mathbf{H}$ is a non-empty collection of non-empty finite sets called *edges*. The *vertices* of $\mathbf{H}$ are the elements of $\bigcup_{H \in \mathbf{H}} H := V(\mathbf{H})$. A *matching* is a collection of mutually disjoint edges, and a *transversal* is a set $T \subset V(\mathbf{H})$ that meets all edges of $\mathbf{H}$. A hypergraph $\mathbf{H}$ is said to be *k-uniform* (for some natural number k) if $|H| = k$ for all $H \in \mathbf{H}$.

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The *matching number*, $\nu(\mathbf{H})$, is the largest number of edges in a matching, and the *transversal number*, $\tau(\mathbf{H})$, is the smallest cardinality of a transversal. It is obvious from the definitions that $\tau(\mathbf{H}) \geq \nu(\mathbf{H})$ and if $\mathbf{H}$ is k-uniform then $\tau(\mathbf{H}) \leq k \cdot \nu(\mathbf{H})$ (because the union of edges in any maximal matching is a transversal). One of the best-known results in graph theory [3] states that every *bipartite graph* $\mathbf{H}$ satisfies $\tau(\mathbf{H}) = \nu(\mathbf{H})$. Taking the terminology from this celebrated theorem, a hypergraph $\mathbf{H}$ is said to have the *König property* if $\tau(\mathbf{H}) = \nu(\mathbf{H})$ holds. For short, such a $\mathbf{H}$ will be called a *K-hypergraph*. Some examples of K-hypergraphs are surveyed in [1].

In this note we investigate the following problem.

Problem 1. *Given a hypergraph $\mathbf{H}$, determine the smallest integer $\kappa = \kappa(\mathbf{H})$ such that $\mathbf{H}$ is the union of κ K-hypergraphs.*

In Theorem 1 of the next section we shall prove that $\kappa(\mathbf{H}) \leq k$ holds for every k-uniform hypergraph $\mathbf{H}$. Though this upper bound is sharp for every k, we expect that it can be improved for some nice classes of hypergraphs.

Example 1. Let $\mathbf{G}$ be (the edge set of) an undirected graph, i.e., a 2-uniform hypergraph. We define a 3-uniform hypergraph called the *triangle hypergraph* or *Δ-hypergraph*, $\mathbf{H}_\Delta(\mathbf{G})$, of $\mathbf{G}$ as follows. The vertices of $\mathbf{H}_\Delta(\mathbf{G})$ are the edges of $\mathbf{G}$, and a 3-tuple is an edge of $\mathbf{H}_\Delta(\mathbf{G})$ if and only if the corresponding three edges of $\mathbf{G}$ form a triangle (a complete graph on three vertices).

One of our long-standing conjectures [5] states

$$\tau(\mathbf{H}_\Delta(\mathbf{G})) \leq 2\nu(\mathbf{H}_\Delta(\mathbf{G})) \qquad (?)$$

for every graph $\mathbf{G}$. So far, this conjecture has only been proved for some particular classes of graphs, see [6]. Here we raise the following closely related problem.

Problem 2. *Does there exist a graph $\mathbf{G}$ with $\kappa(\mathbf{H}_\Delta(\mathbf{G})) = 3$?*

Since $\mathbf{H}_\Delta(\mathbf{G})$ is 3-uniform, κ cannot be larger than 3 (by Theorem 1 below). If $\kappa(\mathbf{H}_\Delta(\mathbf{G})) = 2$ holds in a graph, however, then we immediately obtain that $\tau(\mathbf{H}_\Delta(\mathbf{G})) \leq 2\nu(\mathbf{H}_\Delta(\mathbf{G}))$ holds as well, since $\nu(\mathbf{H}_\Delta(\mathbf{G}')) \leq \nu(\mathbf{H}_\Delta(\mathbf{G}))$ for every $\mathbf{G}' \subset \mathbf{G}$. Therefore, though we expect a *negative* answer to Problem 2, such a result seems to be rather hard to prove at present. On the other hand, if the answer to Problem 2 were affirmative, then it would lead to the question of characterizing the class of graphs $\mathbf{G}$ with $\kappa(\mathbf{H}_\Delta(\mathbf{G})) = 3$.

A stronger variant of Problem 1 is

Problem 3. *Given a hypergraph* $\mathbf{H}$, *determine the smallest integer* $\kappa^* = \kappa^*(\mathbf{H})$ *such that* $\mathbf{H}$ *is the edge-disjoint union of* κ^* *K-hypergraphs.*

Concerning this problem we can only prove that for every k and every k-uniform hypergraph $\mathbf{H}$, $\kappa^*(\mathbf{H}) \leq c \log \tau(\mathbf{H}) \leq c \log(k \cdot \nu(\mathbf{H}))$ for some constant $c < k$ (Theorem 2).

Problem 4. *Let* k *be fixed. Determine* $\kappa^*(k) := \sup \kappa^*(\mathbf{H})$, *where the supremum is taken over all* k*-uniform hypergraphs* $\mathbf{H}$. *In particular, is* $\kappa^*(k)$ *finite for every* k*?*

Let us introduce one more related concept, the hereditary variant of the König property. A hypergraph $\mathbf{H}$ is called *normal* if each of its subhypergraphs (including $\mathbf{H}$ itself) is a K-hypergraph. (Alternatively, normal hypergraphs can be defined in terms of edge colorings, too; i.e., by the equality of chromatic index and maximum degree in every subhypergraph. According to a theorem of Lovász [4], the hereditary edge coloring property is equivalent to the hereditary König property.)

An infinite family of such nice structures is described in the forthcoming paper [8] where it is proved that a variant of the triangle hypergraph for directed graphs — namely, the 3-uniform hypergraph whose edges are the sets of cyclic triangles — is normal in every oriented planar graph.

Now an analogue of Problems 1 and 3 is

Problem 5. *Given a hypergraph* $\mathbf{H}$, *determine the smallest integer* $\kappa^{**} = \kappa^{**}(\mathbf{H})$ *such that* $\mathbf{H}$ *is the union of* κ^{**} *normal hypergraphs.*

In Section 2 we present some estimates on κ^{**}, showing that its value can tend to infinity with the number of vertices. The following problem, however, remains unsolved. We denote by $[\mathbf{n}]^k$ the complete k-uniform hypergraph on n vertices, i.e., the collection of all k-subsets of an n-set.

Problem 6. *Prove that* $\kappa^{**}([\mathbf{n}]^k) = n - k + 1$ *for every* $n \geq k \geq 3$.

A closely related conjecture is published in [7] for hypergraphs $\mathbf{H}$ in which $\nu(\mathbf{H}') = 1$ implies $\tau(\mathbf{H}') = 1$ for every $\mathbf{H}' \subset \mathbf{H}$.

Let us note that every subhypergraph of a normal hypergraph is normal, therefore $\kappa^{**}([\mathbf{n}]^k)$ corresponds to the "worst case" in hypergraphs of order n. (This is also the reason why Problem 5 does not have an "edge-disjoint" version, contrary to Problems 1 and 3.) On the other hand, the König property itself is not hereditary, and it would certainly be an essential step towards the solutions to Problems 2 and 4 if we could reduce the number of candidates for extremal structures in those cases.

2. Results

For uniform hypergraphs, we have the following general upper bound on κ

Theorem 1. *For every k-uniform hypergraph $\mathbf{H}$, $\kappa(\mathbf{H}) \leq k$. Moreover, this upper bound is best possible in the sense that for every integer $k \geq 1$ and $\nu \geq 1$ there is a k-uniform $\mathbf{H}$ with $\kappa(\mathbf{H}) = k$ and $\nu(\mathbf{H}) = \nu$.*

Proof. Let $\{H_1, \ldots, H_\nu\}$ be a maximum matching in $\mathbf{H}$. Denote by $z(i,j)$ $(1 \leq i \leq \nu,\ 1 \leq j \leq k)$ the vertices of H_i. For $1 \leq j \leq k$, set $\mathbf{H}_j = \{H \in \mathbf{H} \mid z(i,j) \in H \text{ for some } i\}$. Since $H_1 \cup \ldots \cup H_\nu$ is a transversal of $\mathbf{H}$ (by the maximality of ν), we have $\mathbf{H}_1 \cup \ldots \cup \mathbf{H}_k = \mathbf{H}$. Moreover, $\{H_1, \ldots, H_\nu\}$ is a matching in every $\mathbf{H}_j$, and $\tau(\mathbf{H}_j) \leq \nu$ also holds since the set $\{z(i,j) \mid 1 \leq i \leq \nu\}$ is a transversal of $\mathbf{H}_j$. Thus, we have obtained a decomposition of $\mathbf{H}$ into k K-hypergraphs, implying $\kappa(\mathbf{H}) \leq k$.

To prove sharpness, we show that $\mathbf{H} = [\mathbf{n}]^k$ with $n = k\nu + k - 1$ satisfies the requirements. The equalities $\nu(\mathbf{H}) = \nu$ and $\tau(\mathbf{H}) = k\nu$ are obvious, and we have already seen that $\kappa(\mathbf{H}) \leq k$. To prove $\kappa(\mathbf{H}) \geq k$, suppose that $\mathbf{H}_1 \cup \ldots \cup \mathbf{H}_\kappa = \mathbf{H}$, where each $\mathbf{H}_i$ $(1 \leq i \leq \kappa)$ is a K-hypergraph. Then $\tau(\mathbf{H}_i) = \nu(\mathbf{H}_j) \leq \nu$, and $k\nu = \tau(\mathbf{H}) \leq \tau(\mathbf{H}_1) + \ldots + \tau(\mathbf{H}_\kappa) \leq \kappa\nu$, because for any choice of transversals T_i in $\mathbf{H}_i$, $T_1 \cup \ldots \cup T_\kappa$ is a transversal of $\mathbf{H}$. Thus, $\kappa \geq k$ as claimed. ■

Theorem 2. *For every k-uniform hypergraph $\mathbf{H}$,*

$$\kappa^*(\mathbf{H}) \leq (\log \tau(\mathbf{H}))/\log(1 + (k-1)^{-1}) < k \log(k \cdot \nu(\mathbf{H})).$$

Proof. Note first that the rightmost inequality is valid by the facts $\tau(\mathbf{H}) \leq k \cdot \nu(\mathbf{H})$ and $e < (1 + (k-1)^{-1})^k$. In order to prove the upper bound of

$(\log\tau(\mathbf{H}))/\log(1+(k-1)^{-1})$, we define two sequences T(0), T(1), T(2), ... of sets and $\mathbf{H}(0)$, $\mathbf{H}(1)$, $\mathbf{H}(2)$,... of hypergraphs with the following properties:

(i) $\mathbf{H}(0)=\mathbf{H}$ and $T(0)$ is a minimum transversal of $\mathbf{H}$ (i.e., $|T(0)| = \tau(\mathbf{H})$);

(ii) $T(i)$ is a transversal of $\mathbf{H}(i)$ whenever $\mathbf{H}(i) \neq \emptyset$;

(iii) $\mathbf{H}(i) \subset \mathbf{H}(i-1)$, and the hypergraph $\mathbf{H}_i := \mathbf{H}(i-1) \setminus \mathbf{H}(i)$ is a K-hypergraph whenever $\mathbf{H}(i-1) \neq \emptyset$;

(iv) $|\mathrm{T}(i)| \leq (1-k^{-1})|\mathrm{T}(i-1)|$ for every i.

If such sequences satisfying properties (i)–(iv) exist, then $(k/(k-1))^u > \tau(\mathbf{H})$ implies $\mathrm{T}(u) = \emptyset$ — i.e., $\mathbf{H}(u) = \emptyset$ — so that $\kappa^*(\mathbf{H}) \leq u$ and the theorem follows. (As a matter of fact, for $\tau \leq k$ we obtain $\kappa^* \leq \tau$ which is much smaller than $k\log\tau$ for k large.)

Suppose that a $\mathbf{H}(i-1) \neq \emptyset$ has been obtained. Take a matching of size $\nu(\mathbf{H}(i-1))$ in $\mathbf{H}(i-1)$, say with edges $H_1, \ldots, H_{\nu(\mathbf{H}(i-1))}$. By property (ii), we can pick a vertex x_j from each $H_j \cap T(i-1)$. Set $X = \{x_j | 1 \leq j \leq \nu(\mathbf{H}(i-1))\}$, $Y = T(i-1) \setminus X$ and $Z = H_1 \cup \ldots \cup H_{\nu(\mathbf{H}(i-1))}) \setminus \mathrm{X}$.

Now we define $\mathbf{H}_i$ as the set of edges of $\mathbf{H}(i-1)$ which meet X, and set $\mathbf{H}(i) = \mathbf{H}(i-1) \setminus \mathbf{H}_i$. Since $\tau(\mathbf{H}_i) \leq |\mathrm{X}| \leq \nu(\mathbf{H}_i) \leq \tau(\mathbf{H}_i)$ (i.e., equality holds throughout), we see that the requirements of (iii) are satisfied. Observe further that Y, as well as Z, is a transversal of $\mathbf{H}(i)$. Hence, it is our free choice to define $T(i) = \mathrm{Y}$ or $T(i) = \mathrm{Z}$, and in either case (ii) will be valid. Trivially, $|\mathrm{Y}| = |T(i-1)| - |\mathrm{X}|$ and since $\mathbf{H}$ is k-uniform $|\mathrm{Z}| = (k-1)|\mathrm{X}|$. Thus, either $|\mathrm{X}| \geq |T(i-1)|/k$ from which we obtain $|\mathrm{Y}| \leq (1-k^{-1})|T(i-1)|$, or else $|\mathrm{X}| < |T(i-1)|/k$ which implies $|\mathrm{Z}| < (1-k^{-1})|T(i-1)|$. Defining $T(i)$ as the smallest of Y and Z, the theorem follows. ■

For normal decompositions of complete hypergraphs, we prove

Theorem 3. *Let* $\geq k \geq 2$.

(i) For $k=2$, $\kappa^{**}([\mathbf{n}]^2) = \lceil \log_2 n \rceil$.

(ii) For $k \geq 3$, $n-k+1 \geq \kappa^{**}([\mathbf{n}]^k) > n/k$.

Proof. (i) Observe that a graph is normal if and only if it is bipartite (since $\tau(C) \neq \nu(C)$ for any odd cycle C). Hence, the assertion is equivalent to the well-know fact that $\lceil \log_2 n \rceil$ is the minimum number of bipartite graphs decomposing a complete graph of order n.

(ii) Let $v_1, v_2, \ldots, v_n$ be the vertices of $[\mathbf{n}]^k$. Define $\mathbf{H}_i$ as the collection of k-tuples H with $v_i \in H$ and $H \cap \{v_j | j < i\} = \emptyset$. Then, for every i

$(1 \le i \le n-k+1)$ and every non-empty $\mathbf{H} \subset \mathbf{H}_i$, $\tau(\mathbf{H}) = \nu(\mathbf{H}) = 1$. Moreover, $\mathbf{H}_1 \cup \ldots \cup \mathbf{H}_{n-k+1} = [\mathbf{n}]^k$, so that $\kappa^{**}([\mathbf{n}]^k) \le n-k+1$.

To prove the lower bound, suppose that $\mathbf{H}_1 \cup \ldots \cup \mathbf{H}_t = [\mathbf{n}]^k$, $t = \kappa^{**}([\mathbf{n}]^k)$, where each $\mathbf{H}_i$ is normal. Then, for every $\mathbf{H}_i$ and every $\mathbf{H} \subset \mathbf{H}_i$, $\nu(\mathbf{H}) = 1$ implies $\tau(\mathbf{H}) = 1$; that is to say, $\mathbf{H}_i$ satisfies the Helly property. By a theorem of Bollobás and Duchet [2] (see also [7]), for $k \ge 3$ the Helly property implies that

$$|\mathbf{H}_i| \le \binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$$

— i.e., $\mathbf{H}_i$ contains at most k/n of the total number of edges in $[\mathbf{n}]^k$ — with equality only if $\mathbf{H}_i$ is the collection $\mathbf{H}(v)$ of all k-element sets containing a fixed vertex v. Thus, if there is an i such that $\mathbf{H}_i \ne \mathbf{H}(v)$ for any v, then $t > n/k$ follows. On the other hand, if $\mathbf{H}_i = \mathbf{H}(v_i)$ for $1 \le i \le t$, then we obtain a much stronger lower bound, namely $t \ge n-k+1$, for otherwise the k-tuple $\{v_{t+1}, v_{t+2}, \ldots, v_{t+k}\}$ would not be an edge of $\mathbf{H}_1 \cup \ldots \cup \mathbf{H}_t$. ∎

In order to improve the lower bound on $\kappa^{**}([\mathbf{n}]^k)$, we would need a better upper bound on the sizes of the $\mathbf{H}_i$. In fact, if $\tau(\mathbf{H}_t) = 1$ holds in a minimal decomposition $\mathbf{H}_1 \cup \ldots \cup \mathbf{H}_t = [\mathbf{n}]^k$ $(t = \kappa^{**}([\mathbf{n}]^k))$, then $\kappa^{**}([\mathbf{n}]^k) \ge \kappa^{**}([\mathbf{n}-1]^k) + 1$ follows, and the lower bound of $n-k+1$ could be proved by induction on n. For this reason, we can assume without loss of generality that $\tau(\mathbf{H}_i) > 1$ for every i. Recently we proved in [9] that in this case

$$|\mathbf{H}_i| \le \binom{n-k-1}{k-1} + \binom{n-2}{k-2} + 1$$

holds whenever n is sufficiently large with respect to k. It may be the case that this inequality is valid for every $n \ge 2k$.

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