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On the Number of Knots and Links

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Ernst and Sumners [2] have shown that if $k(n)$ is the number of prime knots with crossing number n then $k(n)$ grows exponentially with n. Their lower bounds for $k(n)$ imply that

$$\liminf_{n\to\infty} k(n)^{1/n} \geq 2.68\,. \tag{1}$$

Here we obtain an upper bound of the same order for the number of prime links, and hence a fortiori for the number of prime knots. We also obtain new lower bounds.

If $l(n)$ denotes the number of prime links with crossing number n, then for $n \geq 6\,, k(n) < l(n)\,.$

We show that

$$4 \leq \liminf_{n\to\infty} l(n)^{1/n} \leq \limsup_{n\to\infty} l(n)^{1/n} \leq \frac{27}{2}. \tag{2}$$

If we restrict attention to the subclass of alternating prime links and let $a(n)$ denote the number of such links with crossing number n then we prove

$$4 \leq \liminf_{n\to\infty} a(n)^{1/n} \leq \limsup_{n\to\infty} a(n)^{1/n} \leq \frac{27}{4}. \tag{3}$$

The knot theory terminology follows Burde and Zieschang [1]. If K is a tame link in S^3 a regular projection is *minimal* if there is no regular projection of K with fewer crossings. This minimum number of crossings is called the *crossing number* of K. An *alternating link* is one which admits a projection in which the crossings occur alternately over and under as one traverses the link. A *knot* is a link having just one component. A link K in S^3 is *prime* if any 2-sphere which meets K transversely in two points, bounds one and only one 3-ball intersecting K in an unknotted spanning arc. A link is *amphicheiral* or *achiral* if its ambient isotopic to its mirror image, otherwise it is *chiral.*

The lower bound (1) is obtained by considering the special class of Montesinos knots. Every Montesinos knot (link) is prime, and Ernst and Sumners [2] show that if $m(n)$ denotes the number of alternating Montesinos knots of n-crossings, chiral pairs counted separately, then for $n \geq 12$,

$$m(n) \geq \sum_{r=12}^{n} f(r) \quad , \tag{4}$$

where

$$f(r) \geq \sum_{k\geq 3}^{\lfloor r/4 \rfloor} \frac{2^{r-2k-1}}{k3^k} \binom{r-3k-1}{k-1} . \tag{5}$$

Taking $r = n$ and $k = n/8$, gives (1).

Proof of (2) and (3)

Let $\mathcal{K}_n$ be the set of mutually nonisotopic links having crossing number n. Thus $l(n) = |\mathcal{K}_n|$. Thus each member L of $\mathcal{K}_n$ has a representation as a link diagram $\mathcal{H}_L$ which has exactly n crossings and there is no link diagram representing L having fewer crossings. The graph $\mathcal{H}_L$ is planar and 4-regular. We now use the idea of Tait to colour the faces of $\mathcal{H}_L$ black and white such that faces with a common edge have different colours and such that the unbounded face is coloured white. Put a vertex in each black face, and for each crossing, connect the vertices in the two black faces incident with the crossing by a signed edge, the sign of the edge depending on the following convention. If, as one looks along the edge towards the crossing which it cuts, the left hand string passes over the right hand string, then the edge is positive, otherwise it is negative.

This gives a signed graph $\mathcal{G}_L = \mathcal{G}(\mathcal{H}_L)$ which is planar, has n edges and which, because the link L is prime, has no separating vertex. Thus $\mathcal{G}_L$ is 2-connected. Hence if $g_2(n)$ denotes the number of nonisomorphic, 2-connected planar graphs with n edges, then $|\mathcal{K}_n| \leq 2^n g_2(n)$, the factor of 2^n coming from the signing of the edges. Now using Stirling's formula and the result of Tutte [7] that as $n \to \infty$,

$$g_2(n) \sim \frac{2(3n-3)!}{n!(2n-1)!} \quad , \tag{6}$$

gives the required upper bound in (2).

To obtain the upper bound in (3), we use the fact that if L is alternating it has a representative black/white graph $\mathcal{G}_L$ as constructed above in which each edge of $\mathcal{G}_L$ has the positive sign. This is done by taking either $\mathcal{G}_L$ or the plane dual $\mathcal{G}_L^*$ as a representative of L. Accordingly we know $a(n) \leq g_2(n)$ and this gives the right hand side in (3).

The lower bound in (2) will follow trivially from a proof of the lower bound in (3) which we now prove.

Schrijver [4] has shown that Tait's flyping conjecture is true for what he describes as well connected links. An alternating link L is *well connected* if the graph $\mathcal{H}_L$ has no 2-edge cut sets and the only 4-edge cut sets are those determined by one vertex of $\mathcal{H}_L$. This means that a link L is well connected if and only if the graph $\mathcal{G}_L$ is 3-vertex -connected (that is has no vertex cut of less than three vertices and has no parallel edges, except if it has only two vertices connected by at most three parallel edges).

Now if a graph is 3-vertex-connected it follows from a classical theorem of Whitney that there can be no 2-isomorphism and hence no "flype" of the associated alternating link. In other words, each 3-connected planar graph must determine a distinct alternating link. Moreover each of these alternating links must be prime. This follows from the result of Menasco [3], which in the words of Thistlethwaite [6, p. 69] states, "knots with alternating presentations are prime if and only if they look as if they are". Thus the number of prime alternating links with n crossings must be at least as large as the number $g_3(n)$ of 3-connected planar graphs with n edges. But using a result of Tutte [7], that c_n, the number of 3-connected rooted maps without multiple joins is given as $n \to \infty$, by

$$c_n \sim \frac{2n^{-\frac{5}{2}}4^n}{243\sqrt{\pi}}, \tag{7}$$

gives with Stirling's formula the required lower bound in (3). Note that although the formula (7) is in terms of *rooted* maps whether structures are rooted or not is immaterial in (3). ∎

We have tried unsuccessfully to prove that $\lim_{n\to\infty} l(n)^{\frac{1}{n}}$ exists; it would follow from (2) if we could show that l was supermultiplicative, namely, that for $m, n \in \mathbb{Z}$,

$$l(m+n) \geq l(m)\, l(n). \tag{8}$$

We conjecture that $k(n)$ and $a(n)$ are also supermultiplicative but cannot prove it.

Determining $k(n)$ and $l(n)$ exactly is an immensely difficult problem. Thistlethwaite [6] has tabulated all prime knots up to 13 crossings and all prime links up to 9 crossings. A summary of his results is given in the following table, where $l_j(n)$ for $j \geq 2$ denotes the number of prime links with j components and crossing number n and with chiral pairs counted as one. Thistlethwaite (private communication) has also computed the number $k^*(n)$ of amphicheiral knots with crossing number n, for $n \leq 13$. These are also listed, as is $k(n)$.

n	2	3	4	5	6	7	8	9	10	11	12	13
$l_1(n)$	0	1	1	2	3	7	21	49	165	552	2176	9988
$l_2(n)$	1	0	1	1	3	8	16	61				
$l_3(n)$	0	0	0	0	3	1	10	21				
$l_4(n)$	0	0	0	0	0	0	3	1				
$l(n)$	1	1	2	3	9	16	50	132				
$k^*(n)$	0	0	1	0	1	0	5	0	13	0	58	0
$k(n)$	0	2	1	4	5	14	37	98	317	1104	4290	19976

In [2], Ernst and Sumners assumed a growth law of the form $k(n) = a2^{bn}$, and with the above data for $7 \le n \le 13$, estimated $b = 1.725$. This would suggest

$$\lim_{n\to\infty} k(n)^{\frac{1}{n}} \sim 3.306\,. \tag{9}$$

This seems a bit low in view of (2), since it is not unreasonable to suggest, assuming both limits exist, that

$$\lim_{n\to\infty} k(n)^{\frac{1}{n}} = \lim_{n\to\infty} l(n)^{\frac{1}{n}}\,. \tag{10}$$

If we assume (8), a standard theorem of subadditive function theory gives

$$\lim_{n\to\infty} l(n)^{\frac{1}{n}} = \sup_n l(n)^{\frac{1}{n}}\,. \tag{11}$$

Using this with the above data gives estimates for the limit which are too low to satisfy the lower bounds in (2) and (3). It suggests that the "explosion" in the number of knots and links is still to be seen, and makes the task of tabulation for higher crossing number appear even more formidable.

For a further discussion of the knot enumeration problem and its applications in chemistry and DNA research we refer to Sumners [5].

Note. A proof of the full flyping conjecture has been announced (recently) by W. Menasco and M. B. Thistlethwaite.

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