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Orientations of Hamiltonian Cycles in Bipartite Digraphs*

A. P. WOJDA and M. WOŹNIAK

ABSTRACT

The main results of the paper is the following

Theorem. *Let $D = (X, Y, A)$ be a bipartite, balanced digraph of order $2n$ and size at least $2n^2 - 2n + 3$. Then D contains an almost symmetric Hamiltonian cycle (i.e. a Hamiltonian cycle in which at least $2n - 1$ arcs are symmetric edges), unless D has a vertex which is not incident to any symmetric edge of D.*

This theorem implies a number of results on cycles in bipartite, balanced digraphs including some recent results of N. Chakroun, M. Manoussakis and Y. Manoussakis.

1. Definitions and auxiliary results

We consider only finite graphs and digraphs without loops, multiple edges and multiple arcs. With some exceptions specified below, we follow the standard notation and terminology of [2].

A bipartite graph $G = (X, Y, E)$ (where $E \subseteq \{ab | (a \in X$ and $b \in Y)$ or $(a \in Y$ and $b \in X)\})$ or digraph $D = (X, Y, A)$ (where $A \subseteq (X \times Y) \cup$

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$(Y \times X))$ is called *balanced* if $|X| = |Y|$. The set $E = E(G)$ is then called the *edge* set of G, while $A = A \subset D$ is the *arc* set of D. $e(G) = |E(G)|$ and $a(D) = |A(D)|$ are the *size* of the graph G or the digraph D, respectively.

The *order* of a graph G or digraph D is the number of its vertices. If in a digraph D vertices x and y are joined by two opposite arcs (x, y) and (y, x), then the two-cycle xy will be called a *symmetric edge* of D.

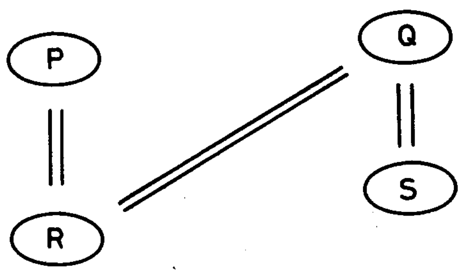
For a digraph D the *underlying graph* $G(D)$ is the graph with the vertex set $V(G(D)) = V(D)$ whose edge set is the set of symmetric edges of D.

The *complement* of a bipartite digraph $D = (X, Y, A)$ is the digraph $\overline{D} = (X, Y, A')$, where $A' = (X \times Y) \cup (Y \times X) - A$.

We denote by $N(x, G)$ the set of neighbours of the vertex x in a graph G. Then the *degree* of x in G is $d(x, G) = |N(x, G)|$. If x is a vertex in a digraph D, then $N^+(x, D) = \{y \in V(D) | (x, y) \in A(D)\}$; $N^-(x, D) = \{y \in V(D) | (y, x) \in A(D)\}$; the *out-degree* $d^+(x, D)$ of x is the cardinality of $N^+(x, D)$, the *in-degree* $d^-(x, D)$ of x is the cardinality of $N^-(x, D)$, while the *degree* of x is $d(x, D) = d^+(x, D) + d^-(x, D)$. A vertex x is called *source* (respectively: *sink*) if $d^-(x, D) = 0$ (respectively $d^+(x, D) = 0$).

For every $n \geq 2$ and $k \leq n/2$ let $G(n, k) = (X, Y, E)$ be the *bipartite, balanced graph of order* $2n$ defined by the following conditions: $X = P \cup Q$, $Y = R \cup S$, $|P| = |R| = k$, $|Q| = |S| = n - k$, $N(a, G(n, k)) = R$ for every $a \in P$ and $d(b, G(n, k)) = n$ for every $b \in Q$ (see Fig. 1).

Observe that the *minimum vertex degree* $\delta(G(n, k))$ is equal to k, $e(G(n, k)) = k^2 + n(n - k)$, and $G(n, k)$ is not Hamiltonian.



$|P| = |R| = k$ $\qquad$ $|Q| = |S| = n - k$

Figure 1.

Let k be an integer, $2 \leq k$. In a digraph D a sequence $(x_1, x_2, \ldots, x_k, x_1)$ of vertices is said to be a cycle of *length* k if for every i and j, $1 \leq i \leq j \leq k$, we have $x_i \neq x_j$, x_i and x_{i+1} are adjacent (the indices in cycles and considered mod k). A cycle in which $x_i x_{i+1}$ is a symmetric edges for every i is *called a symmetric* cycle of D, while a cycle in which all but at most one i, $x_i x_{i+1}$ is a symmetric edge of D is called *almost symmetric*.

A sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$, where $\varepsilon_i \in \{-1, 1\}$, $1 \leq i \leq k$ is called the *orientation* of a cycle $C = (x_1, \ldots x_k, x_1)$ of D if $\varepsilon_i = 1$ implies $(x_i, x_{i+1}) \in A(D)$ and $\varepsilon_i = -1$ implies $(x_{i+1}, x_i) \in A(D)$ for every $i(\text{mod } k)$. Then C is a *realization of* ε in D. Any realization in a digraph D of order n, of the orientation $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, is called a *strong Hamiltonian cycle* if $\varepsilon_i \varepsilon_{i+1} = 1$, and an *antidirected Hamiltonian cycle* if $\varepsilon_i \varepsilon_{i+1} = -1$, for every $i(\text{mod } n)$.

D is *Hamiltonian* if it contains a strong Hamiltonian cycle.

Clearly, if D contains an almost symmetric cycle of length l, then it has every orientation of a cycle of length l.

In this paper we deal with the following problem: for any integer n, find the minimum integer $f(n)$, such that, with some exceptions which may be easily characterized, every balanced, bipartite digraph D of order $2n$ and size at least $f(n)$ has oriented cycles of all even lengths not exceeding the order of G. The corresponding problem for digraphs has been considered by the first author in [7].

We shall need the following result of Moon and Moser [6]:

Theorem A. *Let $G = (X, Y, E)$ be a balanced bipartite graph of order $2n$ and let $S_k = \{x \in X | d(x, G) \leq k\}$ and $T_k = \{y \in Y | d(y, G) \leq k\}$. If for every integer k such that $1 \leq k \leq n/2$, the cardinalities of S_k and T_k are smaller than k then G is Hamiltonian.*

The theorem given below is given in [1] as an exercise.

Theorem B. *Let g be a balanced, bipartite graph of order $2n \geq 4$, such that $\delta(G) \geq k$ and $e(G) > n^2 - k(n - k)$. Then G is Hamiltonian.*

The next theorem we shall use in this paper has been proved by Schmeichel and Mitchem in [5].

Theorem C. *Every balanced and bipartite graph of order $2n$ and size at least $n^2/2$ which is Hamiltonian is also bipancyclic (i.e. contains cycles of all even lengths).*

2. Results

The Theorem B may be improved in the following way:

Theorem 1. *let $G(X,Y,E)$ be a balanced, bipartite graph of order $2n \geq 4$, such that $\delta(G) \geq k$ and $e(G) \geq n^2 - k(n-k)$. Then G is Hamiltonian unless $k \leq n/2$ and G is isomorphic to the graph $G(n,k)$.*

Proof. By contradiction. Let us suppose that G is a non-Hamiltonian graph of order $2n \geq 4$, satisfying the assumptions of Theorem 1 and with the largest number of edges. Then, by Theorem A, in one of the sets of bipartition of $V(G)$, in X say, there are $k' \leq n/2$ vertices of degree at most k'. We have clearly $n^2 - k(n-k) \leq e(G) \leq k'^2 + n(n-k')\ n = n^2 - k'(n-k')$ and, since $\delta(G) \geq k$, $k' \geq k$. The function $f(n,k) = n^2 - k(n-k)$ is decreasing with respect to k when $k \leq n/2$. So $k = k'$, and there exists in X exactly k vertices of degree k and $n-k$ vertices of degree n. Hence $d(y,G) \geq n-k$ for every $y \in Y$.

Let $x \in X$ and $y \in Y$ be two non-adjacent vertices. Then $d(x,G) = k$ and, by the maximality of G, there is in G a Hamiltonian path P with end-vertices x and y, $P = x_1y_1x_2y_2\dots x_ny_n$, $x_1 = x$, $y_n = y$, say; $x_i \in X$ and $y_i \in Y$ for $i = 1,\dots,n$. If $x_1y_i \in E(G)$ then $y_nx_i \notin E(G)$, otherwise $x_1y_1x_2y_2\dots x_iy_nx_ny_{n-1}x_{n-1}\dots y_ix_1$ is a Hamiltonian cycle in G. So we have $d(y,G) = n-k$. Moreover, for every j for which $y_1x_j \in E(G)$, we have $xy_j \notin E(G)$ and therefore $d(y_j,G) = n-k$.

Thus there are in Y $n-k$ vertices of degree $n-k$ non-adjacent to x. Since $e(G) = kn + (n-k)^2$, the remaining k vertices of Y have their degrees equal to n, and therefore G is isomorphic to $G(n,k)$. ■

The following is a simple but useful observation.

Lemma 2. *Let D' be a digraph obtained from D by addition of at most one arc. If $G(D')$ has a cycle of length k, then D contains an almost symmetric cycle of length k.* ■

Now we give our main result.

Theorem 3. *Let $D = (X,Y,A)$ be a balanced, bipartite digraph of order $2n$ and size at least $2n^2 - 2n + 3$. Then D contains an almost symmetric Hamiltonian cycle unless there is in D a vertex which is not incident to any symmetric edge of D.*

Proof. Let D be a digraph satisfying the assumptions of the theorem. We may assume that $\delta(G(D)) \geq 1$. Since $a(\overline{D}) \leq 2n - 3$, we have clearly $e(G(D)) \geq n^2 - 2n + 3$.

We may prove a lemma which will facilitate the proof of Theorem 3.

Lemma 4. *Let $D^+ = (X, Y, A^+)$ be a digraph obtained from the digraph D by addition of an arc (u, v), such that $(u, v) \in A(\overline{D})$. If $\delta(G(D^+)) \geq 2$, then there is in D an almost symmetric Hamiltonian cycle.*

Proof of Lemma. If $G(D^+)$ has a Hamiltonian cycle, then we apply Lemma 2. So let us suppose that $G(D^+)$ is not Hamiltonian.

Since $a(D^+) \geq 2n^2 - 2n + 4$, we have $e(G(D^+)) \geq n^2 - 2n + 4 = n^2 - 2(n - 2)$ and, by Theorem 1, $e(G(D^+)) = n^2 - 2(n - 2)$ and $G(D^+)$ is isomorphic to $G(n, 2)$. Let x_1 and x_2 be the vertices of degree 2 in $G(D^+)$. Without loss of generality we may assume that $x_1, x_2 \in X$. Then $d(x_i, D^+) \leq n + 2$ for $i = 1, 2$, $d(x, D^+) = 2n$ for $x \in X - \{x_1, x_2\}$ and therefore $e(D^+) = d(x_1, D^+) + d(x_2, D^+) + \sum_{x \in X - \{x_1, x_2\}} d(x, D^+) \leq 2(n + 2) + (n - 2)2n = 2n^2 - 2n + 4$. Thus $d(x_1, D^+) = d(x_2, D^+) = n + 2$. The reader may now easily find an almost symmetric Hamiltonian cycle C in D^+ in which the vertices u and v are not consecutive. Therefore C is an almost symmetric cycle in D and the proof of Lemma 4 is complete. ■

By the Lemma 4 we may assume that $\delta(G(D)) = 1$. Let x be a vertex for which $d(x, G(D)) = 1$. Without loss of generality we may assume that $x \in X$. If there is a vertex y, such that $y \neq x$ and $d(y, (G(D)) = 1$, then $y \in Y$ and $xy \notin E(G(D))$, otherwise $a(D) < 2n^2 - 2n + 3$. Similarly we deduce that $d(x, D) = d(y, D) = n + 1$. The digraph D' obtained from D by the addition of the missing arc between x and y satisfies the assumptions of Lemma 4, thus the theorem is proved in this case.

So we may suppose that x is the only vertex of degree 1 in $G(D)$. Let $z \in Y$ be a vertex adjacent to x, but such that $xz \notin E(G(D))$ (such vertex z exists in Y, since otherwise $e(\overline{D}) \geq 2(n - 1)$). The digraph D'' obtained from D by the addition of the missing arc between x and z satisfies the assumptions of Lemma 4. ■

Let $B_1(n) = (X, Y, A)$ be the balanced, bipartite digraph of order $2n$ and size $2n^2 - n$, such that there exists a vertex $x_0 \in X$ for which $d(x_0, B_1(n)) = d^+(x_0, B_1(n)) = n$ and $d(x, B_1(n)) = 2n$ for all remaining vertices of the set X. $B_1'(n)$ is obtained from $B_1(n)$ by reversing all its arcs.

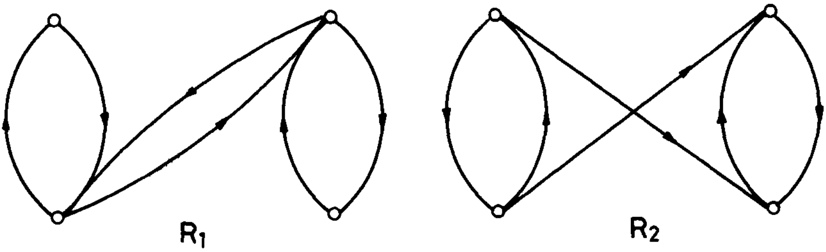


Figure 2.

The digraphs R_1 and R_2 are depicted in Fig. 2.

Corollary 5. [3,4] *Let D be a balanced, bipartite digraph of order $2n \geq 4$ and size at least $2n^2 - n$. Then D is Hamiltonian unless it is isomorphic to one of the digraphs R_1, R_2, $B_1(n)$, $B'_1(n)$.*

Proof. Observe that $2n^2 - n \geq 2n^2 - 2n + 3$ for $n \geq 3$, and then the corollary follows immediately from Theorem 3. The reader may check that it holds also for $n = 2$. ∎

Corollary 6. *Let $D = (X, Y, A)$ be a balanced, bipartite digraph of order $2n \geq 6$ and size at least $2n^2 - 2n + 3$. Then D contains an almost symmetric cycle of every even length $2k$, $4 \leq 2k \leq 2n - 2$.*

Proof. Suppose first that D contains an almost symmetric Hamiltonian cycle. Then there are two vertices x and y, $x \in X$ and $y \in Y$ say, such that the bipartite graph $G(D + (x, y))$ is Hamiltonian (where the digraph $D + (x, y)$ is obtained from D by the addition of the arc (x, y)). Since $e(G(D + (x, y)) > n^2 - 2n + 3 > n^2/2$, $G(D + (x, y))$ is bipancyclic, by Theorem C. So let us assume that D does not contain any almost symmetric Hamiltonian cycle. Then, by Theorem 3, there is in D a vertex x, $x \in X$ say, such that $d(x, D) \leq n$. Let y be a vertex of Y. The bipartite digraph $D - \{x, y\}$ of order $2(n - 1)$ is balanced and $a(D - \{x, y\}) \geq 2n^2 - 2n + 3 - n - 2(n - 1) = 2(n - 1)^2 - n + 3$. Obviously $\delta(G(D - \{x, y\})) \geq 2$ and $e(G(D - \{x, y\})) \geq (n - 1)^2 - n + 3$.

Since $n \geq 3$, we have $e(G(D - \{x, y\})) \geq (n - 1)(n - 3) + 4$ and therefore, by Theorem 1, $G(D - \{x, y\})$ is Hamiltonian. Hence, by Theorem C, $G(D - \{x, y\})$ is bipancyclic and Corollary 6 follows. ∎

N. Chakroun, M. Manoussakis and Y. Manoussakis proved in [4] that every balanced, bipartite digraph of order $2n$ and size at least $2n^2 - 2n + 3$ contains anticycles of every even length. Our next corollary improves this result.

Corollary 7. *Let $D = (X, Y, A)$ be a balanced, bipartite digraph of order $2n$ and size at least $2n^2 - 2n + 3$. Then D contains every orientation of a cycle of any even length $4, 6, \ldots, 2n$, except, possibly, the strong Hamiltonian cycle when D has a sink or a source.*

Proof. By Corollary 6 every bipartite, balanced digraph D of order 2 and size at least $2n^2 - 2n + 3$ contains every orientation of a cycle of length $4, 6, \ldots, 2n - 2$. Let us suppose that there is an orientation of Hamiltonian cycle which is not contained in D. Then, by Theorem 3, there is in D a vertex, x_0 say, such that x_0 is not incident to any symmetric edge of D. We have clearly $d(x_0, D) \geq 3$. Without loss of generality we may assume that $x_0 \in X$. Let a and b be two neighbours of x_0, and let D' be the digraph obtained by the addition to D of the two missing arcs between x and the set $\{a, b\}$.

We check that $\delta(G(D')) \geq 2$ and, by Theorem 1, $G(D'))$ is Hamiltonian. Moreover, we may choose such a and b that either $(a, x_0) \in A(D)$ and $(b, x_0) \in A(D)$ or $(x_0, a) \in A(D)$ and $(x_0, b) \in A(D)$. Hence D contains every non-strong orientation of a Hamiltonian cycle. The only orientation which may be not contained in D is the strong one, and if it happens then either x_0 is a source or a sink. ■

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A. Paweł Wojda
Instytut Matematyki
Akademia Górniczo—Hutnicza
al. Mickiewicza 30
30–059 Kraków, Poland

Mariusz Woźniak
Instytut Matematyki
Akademia Górniczo—Hutnicza
al. Mickiewicza 30
30–059 Kraków, Poland