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A Note on Embedding Graphs Without Small Cycles

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ABSTRACT

Denote by T_k the following statement:

If a graph G is not a star and has no cycles of length i, $3 \leq i \leq k$, as subgraphs, then G is embeddable in its complement.

R. J. Faudree, C. C. Rousseau, R. H. Schelp and S. Schuster have conjectured that T_4 holds.

The paper contains the proof of T_7.

1. Introduction

We shall use standard graph theory notation. A finite, undirected graph G consists of a vertex set $V(G)$ and edge set $E(G)$. All graphs will be assumed to have neither loops nor multiple edges. For graphs G and H we denote by $G \cup H$ the vertex disjoint union of graphs G and H and kG stands for the disjoint union of k copies of graph G. An *embedding* of a graph G (into its complement $\overline{G}$) is a bijection σ on $V(G)$ such that if an edge xy belongs to $E(G)$ then $\sigma(x)\sigma(y)$ does not belong to $E(G)$.

The following theorem was proved, independently, in [2], [3] and [6].

Theorem 1. *Let $G = (V, E)$ be a graph of order n. If $|E(G)| \leq n-2$ then G can be embedded in its complement $\overline{G}$.*

The example of the star $K_{1,n-1}$ shows that Theorem 1 cannot be improved by raising the size of G.

The next theorem completely characterizes those graphs with n vertices and $n-1$ edges which are embeddable ([4,5]).

Theorem 2. *Let G be a graph. If $|E(G)| \leq |V(G)| - 1$ then G is not embeddable if and only if it is isomorphic to one of the following graphs:* $K_{1,p}$, $K_{1,p+3} \cup K_3$ $(p \geq 1)$, $K_1 \cup 2K_3$, $K_1 \cup C_4$, $K_1 \cup K_3$ $K_2 \cup K_3$.

A similar characterization for graphs of order and size equal to n is given in [5]. Also in [5] the authors have remarked that all non-embeddable graphs (with n vertices and no more than n edges) are either stars or contain K_3 or C_4 as subgraphs. For this reason they have conjectured that:

Conjecture 3. *Each non-star graph which contains no cycles of length 3 or 4 as subgraphs is always embeddable.*

The following theorem [5] provides some evidence that the above conjecture might hold.

Theorem 4. *If a graph G with n vertices is not a star, contains no more than $(6/5)n-2$ edges, and has no cycles of length 3 or 4 as subgraphs, then G is embeddable.*

Our purpose is to prove the following

Theorem 5. *If a graph G is not a star and contains no cycles of length 3, 4, 5, 6 or 7 as subgraphs, then G is embeddable.*

The proof of Theorem 5 is given in Section 3. In Section 2 we consider some special cases.

Remark. Theorems 1 and 2 have been improved in many ways. We refer the reader to [1,9] and [10] (cf. also [7] and [8]).

2. Some lemmas

Let G be a connected graph with $\operatorname{diam}(G) = d$ and let A be a subset of $\{0, 1, 2, \dots, d\}$. A permutation σ on $V(G)$ is said to belong to the *class* $\mathcal{P}(G, A)$ iff for every $x \in V(G)$ $\operatorname{dist}_G(x, \sigma(x)) \in A$.

Note that if $\sigma \in \mathcal{P}(G, A)$ and $0 \notin A$ then σ has no fixed point.

Lemma 6. *Let P_n be a path of order n, $n > 3$. Then there exists an embedding of P_n belonging to $\mathcal{P}(P_n, \{1, 2\})$.*

Proof. The fact that Lemma 6 is true in the cases where $n = 4, 5, 6, 7$ is easy to see and can be left to the reader.

For $n > 7$ observe that there exists an edge of P_n, say e, such that $P_n - e$ has two components P_i, P_j with $i, j > 3$. Then, by induction, there exist the embeddings σ_i of P_i and σ_j of P_j such that $\sigma_i \in \mathcal{P}(P_i, \{1, 2\})$ and $\sigma_j \in \mathcal{P}(P_j, \{1, 2\})$. It is easy to see that the permutation on $V(P_n)$ which extends both σ_i and σ_j belongs to $\mathcal{P}(P_n, \{1, 2\})$ and, since σ_i, σ_j have no fixed points, σ is also an embedding of P_n. ■

Lemma 7. *Let G' be a connected graph, $a \in V(G')$ and let G be a graph obtained from G' by adding $k + k'$ new vertices $x_1, \dots, x_k, y_1, \dots, y_{k'}$ and $k + k'$ new edges constituting two paths $ax_1 \dots x_k$ and $ay_1 \dots y_{k'}$ of length k and k', respectively, and having the vertex a as a common end vertex, $1 \le k, k' \le 3$. Suppose there exists an embedding σ' of G' such that $\sigma' \in \mathcal{P}(G', \{1, 2, 3\})$. Then there exists an embedding σ of G such that $\sigma \in \mathcal{P}(G, \{1, 2, 3\})$.*

Proof. The proof is by case by case examination. Let us give two examples, the other cases are left to the reader.

If $k = k' = 1$ we define σ as follows: $\sigma(x_1) = y_1$, $\sigma(y_1) = x_1$ and $\sigma(x) = \sigma'(x)$ for $x \in V(G')$. Since $\operatorname{dist}_G(x_1, y_1) = 2$ so $\sigma \in \mathcal{P}(G, \{1, 2, 3\})$. On the other hand σ is an embedding of G because $\sigma'(a)$ is different from a.

If $k = k' = 3$ we put $\sigma(x_1) = x_3$, $\sigma(x_2) = y_1$, $\sigma(x_3) = x_2$, $\sigma(y_1) = y_3$, $\sigma(y_2) = x_1$, $\sigma(y_3) = y_2$ and $\sigma(x) = \sigma'(x)$ for $x \in V(G')$.

Lemma 8. *If T is a tree of order n and T is not a star then there is an embedding σ of T such that $\sigma \in \mathcal{P}(G, \{1, 2, 3\})$.*

Proof. For $n < 4$ each tree is a star so let $n \geq 4$. For $n = 4$ there is only one tree which is not a star, namely the path P_4, so our lemma holds by Lemma 6. Suppose now it holds for $n' < n$.

If $\text{diam}(T) \geq 7$ then there exists an edge, say e, such that $T - e$ has two components T', T'' which are not stars. The embedding of T belonging to $\mathcal{P}(T, \{1, 2, 3\})$ can be easily obtained from corresponding embeddings for T' and T'' since they do not have any fixed points.

So we may assume that $\text{diam}(T) \leq 6$. Observe that if $\text{diam}(T) = 5$ or 6 then always either Lemma 6 or Lemma 7 can by applied.

Consider now the case where $\text{diam}(T) = 4$. Let $x_1, \ldots, x_5$ be the longest path of T. Observe that if there is a vertex of $T - \{x_1, \ldots, x_5\}$ adjacent to x_2 or x_4 then we could apply Lemma 7. It is easy to see that there are only two cases where neither Lemma 6 nor Lemma 7 can be used. The first one is the graph obtained from the path $x_1, \ldots, x_5$ by adding two new vertices y_1 and y_2 and two edges x_3y_1 an y_1y_2. The second graph is obtained from the path $x_1, \ldots, x_5$ by adding one new vertex connected by an edge with x_3. In the first case the permutation σ can be defined as follows $\sigma(x_1) = x_3$, $\sigma(x_2) = x_1$, $\sigma(x_3) = x_5$, $\sigma(x_4) = y_1$, $\sigma(x_5) = x_2, \sigma(y_1) = y_2, \sigma(y_2) = x_4$.

The second case as well the case where $\text{diam}(T) = 3$ is left to the reader.

■

3. Proof of Theorem 5.

The proof is by induction on n. We may suppose that $n > 8$. We shall distinguish two cases.

Case 1. G is a connected graph. In this case we shall prove that there exists an embedding σ of G such that $\sigma \in \mathcal{P}(G, \{1, 2, 3\})$. By Lemma 8 we can suppose that G is not a tree.

Let $abcd$ be a path of length 3 contained in a cycle of G. Suppose that $G' = G - \{a, b, c, d\}$ has a component which is a star, S say, and let xy be an edge of G connecting the path $abcd$ and S, $x \in \{a, b, c, d\}$. Remark that there is only one such edge since G does not contain a cycle of length less than eight. We can assume that either $S = K_1$ or $S = P_2 = yy_1$ or $S = P_3 = yy_1y_2$. Indeed, otherwise we could use Lemma 7. This implies, by the same argument, that there are no two stars that are components of G' and are connected to the same vertex on the path $abcd$.

If both vertices b and c are connected with two stars S' and S'', components of G', then we replace the path $abcd$ by the path P' with vertex set $V(S') \cup \{b, c\} \cup V(S'')$. Observe that no component of the graph $G'' = G - P'$ is a star since a and d are connected by a path.

If only the vertex b is connected by an edge with S' which is a star component of G' we replace the path $abcd$ by a path with vertex set $V(S') \cup \{b, c, d\} \cup V(S'')$ where S'' denotes a star component of G' connected with d (if it exists). Similarly we proceed if there is no star as components of G' connected by an edge with $\{b, c\}$.

Thus we have showed that it is always possible to choose a path P of length greater than two in such a way that $G' = G - P$ has no component which would be a star.

Denote by G_i, $i = 1, \ldots, k$ the components of G' and by σ_i the embeddings of G_i belonging to $\mathcal{P}(G_i, \{1, 2, 3\})$. Let σ_0 be an embedding of P belonging to $\mathcal{P}(P, \{1, 2, \})$ (cf. Lemma 6). We put $\sigma(x) = \sigma_i(x)$ for $x \in V(G_i)$ and $\sigma(x) = \sigma_0(x)$ for $x \in V(P)$. Evidently $\sigma \in \mathcal{P}(G, \{1, 2, 3\}$.

Suppose σ is not an embedding of the graph G. Thus, by definition of σ, there exist two edges of G, say xy and $x'y'$, such that $\sigma(x)\sigma(y)$=x'y' and $x, x' \in V(P)$, $y, y' \in V(G_i)$ for some i.

Since $\sigma_0 \in \mathcal{P}(P, \{1, 2, \})$ and $\sigma_i \in \mathcal{P}(G_i, \{1, 2, 3\})$ there exists a path from x to x' contained in P of length ≤ 2 and a path from y to y' contained in G_i of length ≤ 3. These paths, together with the edges xx' and yy' constitute a cycle of length less or equal to 7, a contradiction.

Case II. Suppose the graph G is not connected. Denote by r the number of components of $G, r \geq 2$. Add $r - 1$ edges to join distinct components. An embedding of the obtained connected graph is also an embedding of G. ■

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