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A 11/8-Approximation Algorithm for the Steiner Problem on Networks with Rectilinear Distance

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ABSTRACT

The rectilinear distance between two points $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ in the plane equals $|x_1 - x_2| + |y_1 - y_2|$ where (x_i, y_i) are the Cartesian coordinates of v_i. The Steiner problem requires a shortest tree spanning a given set of n distinguished points. An 11/8-approximation algorithm for this problem is given. The approximate Steiner tree can be computed in time $O(n^3)$.

1. Introduction

The rectilinear distance between two points $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ in the plane equals $|x_1 - x_2| + |y_1 - y_2|$. *The Steiner problem on networks with rectilinear distance,* which is NP-complete (see [3]), requires a shortest tree, $T_{\min}$, spanning a given set of distinguished points S. This problem may be considered as the Steiner problem on the following distance graph G: the vertices of G coincide with the intersections of lines which are parallel to axes and contain the S-points; the edges of G coincide with segments of these lines.

We will use the following notations. The *metric closure* of G is a complete graph $\overline{G} = (V, \overline{E}, \overline{d})$ which has edge lengths equal to shortest path distances in G. G_M is the subgraph of G induced by a vertex subset

$M \subseteq V$. In particular, $\overline{G}_S$ is the subgraph of $\overline{G}$ induced by S. Let us denote by $d(A)$ the length of A and by $\text{mst}(G)$ the length of a minimum spanning tree (MST) of G. The famous MST-algorithm approximates $T_{\min}$ with an MST of $\overline{G}_S$ with edges replaced by the corresponding shortest paths in G. Hwang [1] proved the following inequality:

$$\text{mst}(\overline{G}_S)/d(T_{\min}) \leq 3/2 \tag{1.1}$$

He proved that 3/2 is the supremum for the left ratio in (1.1) and presented a faster implementation of the MST-algorithm which computes the approximate tree in time $O(|S| \log |S|)$ [2].

In this paper, we present the 11/8–approximation algorithm with the implementation time $O(|S|^3)$. This algorithm improves the MST-algorithm by taking into consideration some additional points outside S.

Some preliminary definitions: Given a metricly closed graph $G = (V, E, d)$, we may *contract* an edge e, i.e. reduce its length to 0. We use $G[e] = (V, E, d')$ to denote the resulting graph. For any triple $z = \{u, v, w\}$ this graph $G[z]$ equals $G[(u, v)][(v, w)]$, i.e. it results from two contractions.

Our algorithm goes as follows:

Algorithm 1.1.

(1) $F \leftarrow \overline{G}_S : W \leftarrow \emptyset$; Triples $\leftarrow \{z \subset S : \#z = 3\}$

(2) For every $z \subset$ triples do

  find v which minimizes $\sum_{s \in z} d(v, s)$

  $v(z) \leftarrow v$

  $d(z) \leftarrow \sum_{s \in z} d(v(z), s)$

(3) repeat forever

  (a) find $z \subset$ *Triples* which maximizes $\text{win} = \text{mst}(F) - \text{mst}(F[z]) - d(z)$

  (b) if $\text{win} \leq 0$ then exit repeat

  (c) $F \leftarrow F[z]$; $\text{insert}(W, v(z))$

(4) Find a Steiner tree T_α for $S \cup W$ in graph G using MST algorithm.

In a few words, the presented algorithm (see step (3) as long as possible finds the best reduction of the of MST of F which initially coincides with $\overline{G}_S$, by adding to an MST of F, three G-edges with a common end (i.e., a *star*) and removing the longest edges from each resulting cycle. After the triple contraction we obtain the next F.

We proved the following approximating bound for the Steiner tree in networks [4]:

$$d(T_\alpha)/d(T_{\min}) \leq 11/6.$$

The main result of this paper is the following

Theorem 1.2. *For networks with rectilinear distance*

$$d(T_\alpha)/d(T_{\min}) \leq 11/8.$$

In the next section we research the quality of the greedy approximation of the best possible MST-length reduction by starts. We show that the approximation ratio does not exceed 2. Section 3 is devoted to the approximation of the minimum rectilinear Steiner tree by the tree which one can obtain by the best possible MST-length reduction by starts. We show that in this case approximation ratio does not exceed 5/4. These two approximation bounds imply the statement of Theorem 1.2. In the last section a faster implementation (in time $O(|S|^3)$) of Algorithm 1.1 is given.

2. The win of a greedy sequence of triples

Let $F = \langle V, E, d_F \rangle$ be a graph and d be a length function defined on the set of triples of vertices. For a set Z of triples, $d(Z)$ is the sum of lengths of its elements.

For a set A consisting of edges and triples we define $F[A]$ recursively: $F[\emptyset] = F$, and $F[A \cup e] = F[A][e]$. (For brevity, we denote a singleton $\{x\}$ as x.) For a set Z of triples, we define $\text{win}_F(Z) = \text{mst}(F) - \text{mst}(F[Z]) - d(Z)$.

Let $z_1, \ldots, z_m$ be a sequence of triples. We say that this sequence is *greedy in F* if it satisfies the following conditions: if $\text{win}_F(Z) \leq 0$ for every triple z, then $m = 0$, otherwise, $\text{win}(z_1) \geq \text{win}(z)$ for every triple z; the sequence $z_2, \ldots, z_m$ is greedy in $F[z_1]$.

In this section we prove the following

Theorem 2.1. *If H is the set of elements of a greedy sequence of triples, then for every set of triples Z*

$$2\text{win}_F(H) \leq \text{win}_F(Z). \tag{2.1}$$

Proof. For an edge e of F define $\text{save}_F(e) = \text{mst}(F) - \text{mst}(F[e])$.

Lemma 2.2. *Let $F_{\leq x}$ be a graph with the same vertices and edges as F, except that the edges of length larger than x are removed. Then $\text{save}_F(e)$ is the minimal value x such that both ends of e are in the same component of $F_{\leq x}$.*

Proof. Let x be defined as above. Consider an MST of $F[e]$ which contains e, say T. $T - e$ has two connected components, each containing one of the ends of e. By definition, there exists a path in F which connects ends of e together, such that all its edges have length at most x. One edge on this path, say a, must cross from one component to the other, thus $T \cup a - e$ is a spanning tree of F with length at most $\text{mst}(F[e]) + x$.

On the other hand, consider T', an MST of F, and the unique path in T' which connects the ends of e. By definition, one edge on this path must have length at least x. Removing this edge from T' and inserting the contracted edge e results in a spanning tree of $F[e]$ with length at most $\text{mst}(F) - x$. ■

Lemma 2.3. *For every set of edges A and any edge b, we have*

$$\text{save}_{F[A]}(b) \leq \text{save}_F(b).$$

Proof. It follows directly from the above characterization, as the set of edges $F[A]_{\leq x}$ contains the set of edges $F_{\leq x}$. ■

Lemma 2.4. *Let Z be a set of triples. Then for every edge e either $\text{win}_{F[e]}(Z) = \text{win}_F(Z)$ or there exists $z \in Z$ such that*

$$\text{win}_{F[e]}(Z - z) \geq \text{win}_{F[z]}(Z - z) \tag{2.2}$$

Proof. Let T be an MST of F. The unique path P in T which connects the ends of e contains an edge a such that $d_F(a) = \text{save}_F(e)$. Let U and W be the two connected components of $T - a$. Consider now T', an MST of $F[Z]$. On the unique path P' in T' connecting the ends of e at least one edge, say b, connects U and W. The choice of b assures $\text{save}_F(b) = \text{save}_F(b)$.

Assume first, that $d_{F[Z]}(b) = d_F(b)$. Then $\text{save}_{F[Z]}(e) \geq d_{F[Z]}(b) \geq \text{save}_F(e)$, consequently $\text{save}_{F[Z]}(e) = \text{save}_F(e) = c$. As a result, $\text{win}_{F(e)}(Z) = \text{mst}(F[e]) - \text{mst}(F[\dot{e} \cup Z]) - d(Z) = [\text{mst}(F) - c] - [\text{mst}(F[Z]) - c] - d(Z) = \text{win}_F(Z)$.

Now we may assume that $d_{F[Z]}(b) < d_F(b)$. In this case $b \subset z$ for some triple $z \in Z$. We will show that such a z satisfies (2.2).

Because $\text{save}_F(b) = \text{save}_F(e)$, we know that $\text{mst}(F[e]) = \text{mst}(F[b])$. This allows us rewrite the left side of (2.2) as follows.

$$\begin{aligned}\text{win}_{F[e]}(Z-z) &= \text{mst}(F[e]) - \text{mst}(F[e \cup Z - z]) - d(Z-z)\\ &= \text{mst}(F[b]) - \text{mst}(F[Z-z][e]) - d(Z-z).\end{aligned}$$

Without loss of generality we may assume that $z = b \cup c$, where c is an edge which belongs to $T' - P'$. Thus removing b and c from T' creates three components, and two of them can be connected either by b or by e. This shows that $\text{save}_{F[Z-z]}(b) = \text{save}_{F[Z-z]}(e)$. We will use this identity to rewrite the right side of (2.2) as follows:

$$\begin{aligned}&\text{win}_{F[z]}(Z-z) = \text{mst}(F[z]) - \text{mst}(F[Z]) - d(Z-z) =\\ &\text{mst}(F) - \text{save}_F(e) - \text{save}_{F[b]}(c) - \text{mst}(F[Z]) - d(Z-z) =\\ &\text{mst}(F[b]) - \text{save}_{F[b]}(c) - \text{mst}(F[Z-z][b][c]) - d(Z-z) =\\ &\text{mst}(F[b]) - \text{save}_{F[b]}(c) - \text{mst}(F[Z-z][b]) + \text{save}_{F[Z-z][b]}(c) - d(Z-z) =\\ &\text{mst}(F[b]) - \text{save}_{F[b]}(c) - \text{mst}(F[Z-z][e]) + \text{save}_{F[Z-z][b]}(c) - d(Z-z).\end{aligned}$$

Thus it suffices to prove that $\text{save}_{F[b]}(c) \geq \text{save}_{F[Z-z][b]}(c)$, which follows immediately from Lemma 2.2. ∎

Now we can prove Theorem 2.1 by induction on $\#H$. If $H = \emptyset$, then $\text{win}_F(Z) \leq 0$. This follows from the fact that in this case for every triple z $\text{win}_F(z) \leq 0$ and $\text{win}_F(Z-z) = \text{win}_F(Z) - \text{win}_F(z)$.

In the inductive step, let h be the first element of a greedy sequence. For some two edges a and b we have $h = a \cup b$. By Lemma 2.4, there exists subset Y of Z with at most two elements, such that

$$\begin{aligned}&\text{win}_{F[z]}(Z) = win_{F[z][x]}(Z) \leq win_{F[x]}(Z-Y) =\\ &\text{win}_F(Z) - win_y(Y) \geq win_z(Z) - 2\text{win}_F(h).\end{aligned}$$

The last inequality follows from trivially if Y is empty or singleton set. If $Y = \{y, z\}$, then by Lemma 2.3

$$\begin{aligned}&\text{win}_F(Y) = \text{win}_F(y) + \text{win}_F(z)\\ &\text{win}_F(y) + \text{save}_{F[y]}(z) - d(z) \leq \text{win}_F(y) + \text{save}_F(z) - d(z) =\\ &\text{win}_F(y) + \text{win}_F(z) \leq 2\text{win}_F(h).\end{aligned}$$

Note that $H - h$ is the set of elements of a greedy sequence in $F[h]$. By inductive hypothesis and inequality $\text{win}_{F[h]}(Z) \geq \text{win}_F(Z) - 2\text{win}_F(h)$, we may conclude that

$$\begin{aligned}&2\text{win}_F(H) = 2\text{win}_{F[h]}(H-h) + 2\text{win}_F(h) \geq\\ &\text{win}_{F[h]}(Z) + 2\text{win}_F(h) \geq \text{win}_F(Z). \;\blacksquare\end{aligned}$$

3. Four sets of stars induced by the Steiner tree

A collection of a triple z with three edges, connecting the elements of z with a vertex v which minimizes $\sum_{s\in z} d(v,s)$, will be called *a* star of z with the *center* $v = v(z)$ and the *length* $d(z) = \sum_{s\in z} d(v(z),s)$.

Lemma 3.1. *Let G be a graph induced by a set of points S. There is a set of stars Z such that*

$$4[\mathrm{mst}(F) - \mathrm{win}_F(Z)] \le 5d(T_{\min}), \tag{3.1}$$

where F denotes the complete graph $\overline{G}_S$.

Proof. Suppose $T_{\min}$ has a cut-vertex $s \in S$ which partitions $T_{\min}$ into two subtrees T_1 and T_2 spanning S_1 and S_2 respectively ($S_1 \cup S_2 = S$, $S_1 \cap S_2 = s$). Moreover suppose Lemma is true for S_i, i.e. there are star sets Z_i for graphs $F_i = \overline{G}_{S_i}$ such that

$$4(\mathrm{mst}(F_i) - \mathrm{win}_{F_i}(Z_i)) < 5d(T_i) \qquad (i = 1, 2).$$

Then the Lemma is true for S:

$$\begin{aligned}
\mathrm{mst}(F) - \mathrm{win}(Z_1 \cup Z_2) &= \mathrm{mst}(F) - \mathrm{win}_F(Z_1) - \mathrm{win}_{F[Z_1]}(Z_2) = \\
= d(Z_1) + \mathrm{mst}(F[Z_1]) - \mathrm{win}_{F[Z_1]}(Z_2) &\le d(Z_1) + \mathrm{mst}(F_2) - \mathrm{win}_{F[Z_1]}(Z_2) = \\
= d(Z_1) + d(Z_2) = \mathrm{mst}(F_1) &- \mathrm{win}_{F_1}(Z_1) + \mathrm{mst}(F_2) - \mathrm{win}_{F_2}(Z_2) \le \\
\le 5/4[d(T_1) + d(T_2)] &= 5/4 d(T_{\min}).
\end{aligned}$$

Thus it is sufficient to consider such instances of the Steiner problem which have not a minimum Steiner tree with cut-vertices belonging to S. In this case, Hwang [1] proved that $T_{\min}$ can be represented for odd $|S|$ by the straight line (for certainty we assume that it is a horizontal) with alternative up-down vertical edges incident to leaves (Fig. 3.1). For even $|S|$ this straight line may have a corner (the dotted edge on the right end) (Fig. 3.1).

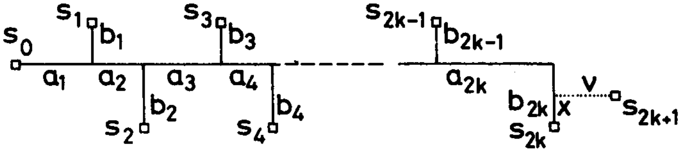

Figure 3.1.

We numerate the leaves and the edges from left to right. The horizontal and vertical edge lengths are denoted by a_i and b_i, respectively. Any vertical edge which contains S-vertex with the odd (even) index will be called odd (even).

Further we use the following denotations. Let T_Z be the Steiner tree induced by a set of stars Z. The *deficit* $\Delta(Z)$ *of* Z equals to the subtraction $\Delta(Z) = d(T_Z) - d(T_{\min})$. Every star will be identified by its three ends belonging to S (or two for degenerate stars). $x \dot{-} y = \begin{cases} x - y, & \text{if } x > y \\ 0, & \text{if } x \le y. \end{cases}$

On the following figures the dotted lines denote parts of star edges which are not coincide with $T_{\min}$-edges, the double lines denote parts of $T_{\min}$-edges with lengths contained in the deficit sum of the star set Z, the asterisks denote the centers of Z-stars.

We shall construct sets of stars Z_i $(i = 1, \ldots, 4)$ for odd $|S|$ by induction on k $(k = 0, 1, \ldots, |S|/2)$ with the following properties:

(1) The star centers of the sets Z_1 and Z_3 (Z_2 and Z_4) lie on the odd (even) verticals.

(2) The sets Z_2 and Z_4 contain the degenerate star (s_0, s_1).

(3) $\sum_{i=1}^{4} \Delta(Z_i) \le d(T_{\min}) - b_{2k}$, if $b_1 \ge b_3$ then $\sum_{i-1}^{4} \Delta(Z_i) \le d(T_{\min}) - b_{2k} - a_2$.

The base of induction: $k = 2$.

$Z_1 = \{(s_0, s_1, s_2), (s_2, s_3, s_4)\}$. $\Delta(Z_1) = b_2 \dot{-} b_4$.

$Z_2 = \{(s_0, s_1), (s_1, s_2, s_3), (s_3, s_4)\}$. $\Delta(Z_2) = \max\{b_1, b_3\}$ (Fig. 3.2).

Let $b_1 \ge b_3$. Then $Z_3 = \{(s_0, s_1, s_3), (s_2, s_3, s_4)\}$, $\Delta(Z_3) = a_3$; $Z_4 = \{(s_0, s_1), (s_0, s_2, s_3), (s_3, s_4)\}$, $\Delta(Z_4) = a_1 + b_3$ (Fig. 3.3).

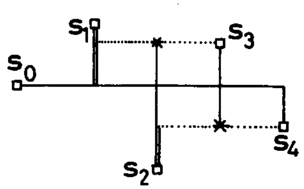

Figure 3.2.

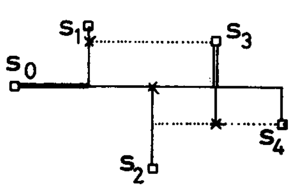

Figure 3.3.

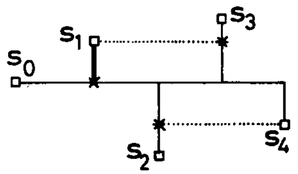

Figure 3.4.

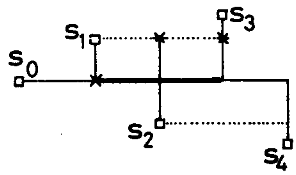

Figure 3.5.

Let $b_1 < b_3$. Then $Z_3\{(s_0, s_1, s_2), (s_1, s_3, s_4)\}$, $\Delta(Z_3) = a_2$. If $b_2 \geq b_4$ then $Z_4 = \{(s_0, s_1), (s_0, s_2, s_3), (s_3, s_4)\}$. $\Delta(Z_4) = b_1 + a_4$ (Fig. 3.4). If $b_2 < b_4$ then $Z_4 = \{(s_0, s_1), (s_1, s_2, s_3), (s_2, s_4)\}$. $\Delta(Z_4) \leq a_3$ (Fig. 3.5).

It is easy to check properties (1)–(3).

The step of induction.

By the induction hypothesis there are star sets Z_i $(i = 1, \ldots, 4)$ satisfying conditions (1)–(3) for double part of the tree $T_{\min}$ (Fig. 3.6). Modifying these sets we obtain star sets Z_i' $(i = 1, \ldots, 4)$ for the whole tree $T_{\min}$.

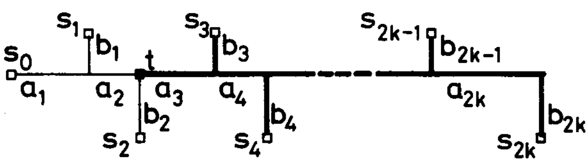

Figure 3.6.

(i) $Z_1' = Z_1 - (t, s_3, s_4) \cup \{(s_0, s_1, s_2), (s_2, s_3, s_4)\}$. $\Delta(Z_1') = \Delta(Z_1) + b_2 \dot{-} b_4$.

(ii) $Z_2' = Z_2 - (t, s_3) \cup \{(s_1, s_2, s_3), (s_0, s_1)\}$. $\Delta(Z_2') = \Delta(Z_2) + b_1 \dot{-} b_0$.

(iii) Let $b_2 \leq b_4$. If $(t, s_3, s_4) \in Z_3$ then

$Z_3' = Z_3 - (t, s_3, s_4) \cup \{(s_2, s_3, s_4), (s_0, s_1, s_2)\}$. $\Delta(x_3') = \Delta(Z_3)$ (Fig. 3.7).

If $(t, s_3, s_5) \in Z_3$ then $Z_3' = Z_3 - ((t, s_3, s_5) \cup \{(s_2, s_3, s_5), (s_0, s_1, s_2)\}$. $\Delta(Z_3') = \Delta(Z_3) + b_2$ (Fig. 3.8).

Let $b_2 > b_4$, z_3 be the Z_3-star with the center on the s_3-vertical. If $b_1 \geq b_3$ then $Z_3' = Z_3 - z_3 \cup \{(s_0, s_1, s_2), (s_2, s_3, s_4)\}$ and $\Delta(Z_3') = \Delta(Z_3) + a_3$

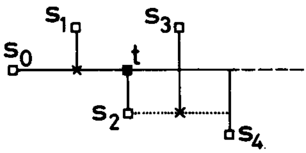

Figure 3.7.

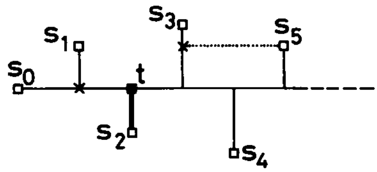

Figure 3.8.

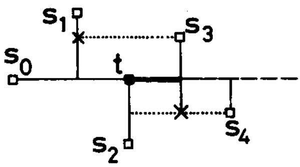

Figure 3.9.

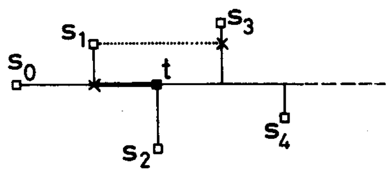

Figure 3.10.

(Fig. 3.9). If $b_1 < b_3$ then $Z'_3 = Z_3 - z_3 \cup \cup\{(s_0, s_1, s_2), (s_1, s_3, s_4)\}$ and $\Delta(Z'_3) = \Delta(Z_3) + a_2$ (Fig. 3.10).

If $z_3 - (t, s_3, s_5)$ then the next Z_3-star will connect s_4 and s_5 and it is indifferent s_4 or s_5 belongs to the previous Z_3-star.

(iv) If $(t, s_3, s_5) \in Z_4$, then $Z'_4 = Z_4 - (t, s_3) \cup \{(s_0, s_1), (s_0, s_2, s_3)\}$. $\Delta(Z'_4) = \Delta(Z_4) + a_1$ (Fig. 3.11).

Let $(t, s_3, s_5) \in Z_4$, $Z''_4 = Z_4 - \{(t, s_3), (t, s_4, s_5)\}$.

If $b_1 \geq b_3$ then $Z'_4 = Z''_4 \cup \{(s_0, s_1), (s_0, s_2, s_3), (s_3, s_4, s_5)\}$ and $\Delta(Z'_4) = \Delta(Z_4) - a_3 + b_3 - b_5$ (Fig. 3.12). Note that $b_3 - b_5$ may be placed into the b_1-vertical part which is not in $\Delta(Z'_i)$, $i = 1, \ldots, 3$.

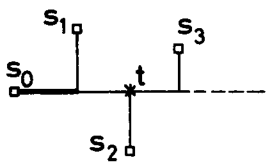

Figure 3.11.

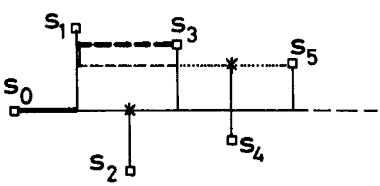

Figure 3.12.

Assume, that $b_1 < b_3$. If $b_3 \leq b_5$ (Fig. 3.13) then
$Z'_4 = Z''_4 \cup \{(s_0, s_1), (s_1, s_2, s_3), (s_3, s_4, s_5)\}$. $\Delta(Z'_4 - \Delta(Z_4) - a_3$.

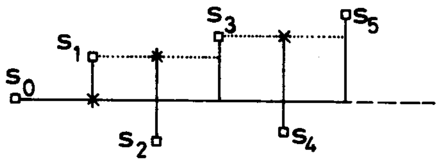

Figure 3.13.

Let $b_5 < b_3$. If $b_4 \leq b_2$ (Fig. 3.14) then $Z_4' = Z_4'' \cup \{(s_0, s_1), (s_1, s_2, s_3), (s_3, s_4, s_5)\}$. $\Delta(Z_4') = \Delta(Z_4) - a_3 + a_4 + b_1$.

If $b_4 > b_2$ then $Z_4' = Z_4'' \cup \{(s_0, s_1), (s_1, s_2, s_3), (s_3, s_4, s_5)\}$. $\Delta(Z_4') = \Delta(Z_4)$ (Fig. 3.15).

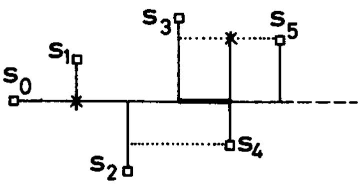

Figure 3.14.

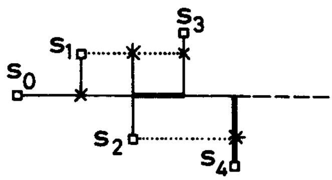

Figure 3.15.

It is easy to check properties (1)–(3).

Thus it is constructed four sets of stars Z_i $i = 1, \dots, 4$ which satisfy (1)–(3).

The property (3) implies for odd $|S|$ that $\sum_{i=1}^{4} \Delta(Z_i) \leq d(T_{\min})$.

Let $|S|$ be even. Replacing in Z_2 and Z_4 degenerate stars (s_{2k-1}, s_{2k}) or (s_{2k-2}, s_{2k}) by stars $(s_{2k-1}, s_{2k}, s_{2k+1})$ or $(s_{2k-2}, s_{2k}, s_{2k+1})$, respectively, we get the sets Z_2' and Z_4' with the same deficits. Adding to Z_1 and Z_3 the degenerate star (s_{2k}, s_{2k+1}) and replacing in Z_1 the star $(s_{2k-2}, s_{2k-1}, s_{2k})$ by the star $(s_{2k-2}, s_{2k}, s_{2k+1})$ we get $\Delta(Z_1') = \Delta(Z_1) + y$ and $\Delta(Z_3') = \Delta(Z_3) + x$ (Fig. 3.16). Therefore the property (3) implies that for even $|S|$ there are four star sets such that $\sum_{i=1}^{4} \Delta(Z_i) \leq d(T_{\min})$.

Thus, there is a star set Z with deficit $\Delta(Z) \leq 1/4\ d(T_{\min})$, i.e. Z induces the Steiner tree T_Z such that

$$d(T_Z) \leq 5/4\ d(T_{\min}).$$

The last inequality implies Lemma 2.4.

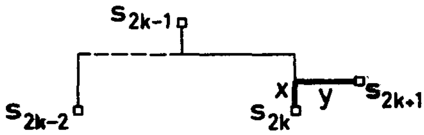

Figure 3.16.

To prove Theorem 1.2 it is enough to note that Algorithm 1.1 finds a greedy sequence of triples H, for which the following sequence of inequalities holds:

$$8d(T_a) = 8\text{mst}(\overline{G}_{S\cup W}) \le 8\text{mst}(\overline{G}_S \cup H) = 8(\text{mst}(\overline{G}_S) - \text{win}(H)) =$$
$$= 4\text{mst}(\overline{G}_S) + 4[\text{mst}(\overline{G}_S) - 2\text{win}(H)] \overset{(2.1)}{\le} 4\text{mst}(\overline{G}_S) + 4[\text{mst}(\overline{G}_S) - \text{win}(Z)] \le$$
$$\overset{(3.1)}{\le} 4\text{mst}(\overline{G}_S) + 5d(T\min) \overset{(1.1)}{\le} 6d(T_{\min}) + 5d(T_{\min}) = 11d(T_{\min}) \ \blacksquare$$

4. A faster approximation algorithm

The following procedure finds the save matrix of a Steiner tree T in time $0(|S|^2)$.

findsave (T)

if $T \neq e$ then

$e \leftarrow$ an edge of T with maximum length; $x \leftarrow d(e)$

$T_1, T_2 \leftarrow$ the connected components of $T - e$

for each vertex v_1 of T_1 and each vertex v_2 of T_2 do save$(v_1, v_2) \leftarrow x$

findsave (T_1); findsave (T_2)

Let $G = (V, E)$ be the graph with rectilinear distance and z be a triple. Then $V(z)$ denotes the set of S-vertices situated between (in the metric sense of this word) the elements of z, $v(z)$ denotes the center of z.

Algorithm 4.1.

(1) $F \leftarrow \overline{G}_S; W \leftarrow \emptyset; P \leftarrow \{z \subset S : \#z = 3 \ \& \ V(z) = z\}$.

(2) repeat forever

(a) $T \leftarrow$ an MST of F; findsave(T)

(b) find $z \in p$ which maximizes

$$\text{win} = \max_{e \subset z} \text{ save}(e) + \min_{e \subset z} \text{ save}(e) - d(z)$$

(c) if win ≤ 0 then exit repeat

(d) $F \leftarrow F[z]$; insert$(W, v(z))$

(3) Find a rectilinear Steiner tree for $S \cup W$ using MST algorithm.

The next lemma shows that algorithms 4.1 and 1.1 give the same tree.

Lemma 4.2. *The greedy sequence of triples belongs to P.*

Proof. The region $V(z)$ for a triple $z = \{s_1, s_2, s_3\}$ is shown on Fig. 4.1. Let $s_0 \in V(z)$. Note that if any edge (s_0, s_i) contracted, then z coincides with the star z' such that $d(z') < d(z)$ and the step (b) will choose z' instead of z.

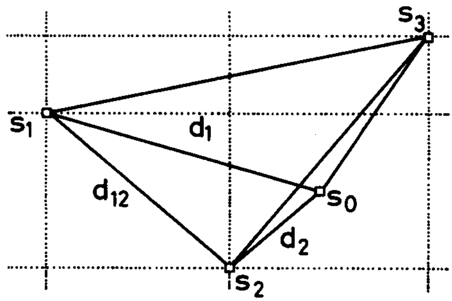

Figure 4.1.

Consider a triangle $\{s_0, s_1, s_2\}$. Denote $e_1 = (s_0, s_1)$, $e_2 = (s_0, s_2)$, $e_3 = (s_1, s_2)$, $d_i = \text{save}_F e_i$, $d_k = \min\{d_i,\ i = 1,2,3\}$. Since $\text{save}_{F[e_k]} e_{k+1} = \text{save}_{F[e_k]} e_{k-1}$ implies $d_{k+1} = d_{k-1}$, There are two alternative cases:

1) $d_1 = d_2 \geq d_3$ or $d_2 \leq d_1 = d_3$. The save values of the triangle $z' = \{s_1, s_3, s_0\}$ are no more than the ones of z but $d(z') \leq d(z)$.

2) $a_1 < a_2 = a_3$. The save values of the triangle $z' = \{s_1, s_2, s_0\}$ are no more than the ones of z but $d(z') < d(z)$. ∎

Lemma 4.3. $|P| < |S|^2$.

Proof. For the sake of simplicity assume that S-vertices have no coinciding Cartesian coordinates. A vertex $v \in V \setminus S$ is called upper (lower) if the S-vertex with the same abscissa is lower (upper) than v (Fig. 4.2).

Let $k(v)$ be a number of P-triangles which have a vertex v as a center. It is clear that for any upper (lower) vertex v there are exactly $k(v)-1$ upper (lower) vertices v' with $k(v') = 0$. So $|P| \leq \sum_{v \in V \setminus S} k(v) \leq |V \setminus S| < |S|^2$ ∎

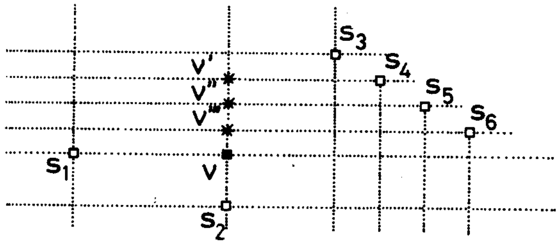

Figure 4.2.

Theorem 4.4. *The implementation time of Algorithm 4.1 is* $0(|S|^3)$.

Proof. The following implementation time bounds for steps of Algorithm 4.1 are obvious: (1) — $0(|S|^3)$; (3) — $0(|S| \log |S|)$ [2]; (a)–(b) — $0(|S|^2)$ (Lemma 4.3) and therefore (2) — $0(|S|^3)$. ■

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