Decomposing random graphs into few cycles and edges

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joint work with Michael Krivelevich and Benny Sudakov

Path decompositions

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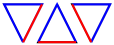

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Dropping to average degree $d/2$ takes at most n cycles.
After removing $O(n \log n)$ cycles, a forest remains. □

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Our result addresses the random graph bound and determines the right asymptotics.

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Let $p = p(n)$ be some probability function. We say that some property P holds for $G(n, p)$ *with high probability* or *whp*, if

$$\lim_{n \to \infty} \mathbf{P}(P \text{ holds for } G(n, p)) = 1.$$

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Altogether, at least $\frac{odd(G(n,p))}{2} + \frac{np}{2}$ cycles and edges.

Our result

Theorem (K–Krivelevich–Sudakov, 2014+)

If $p \gg \frac{\log \log n}{n}$ *then* whp, $G(n, p)$ *can be decomposed into*

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cycles and edges.

Remark. In most of the probability range, $odd(G(n,p)) \sim n/2$.

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1. Remove edges to obtain an Euler graph.

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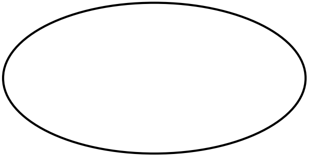
Plan:

1. Remove edges to obtain an Euler graph.
2. Then remove long cycles to get an Euler graph on linearly many edges.
3. Break it up into cycles arbitrarily.

The odd-degree vertices

odd vertices



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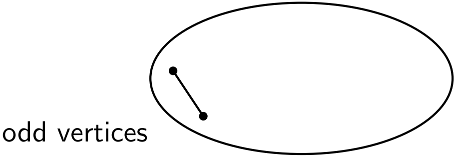

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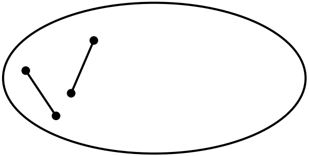

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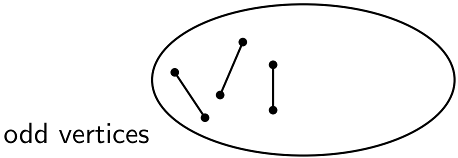

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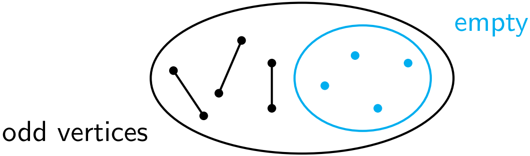

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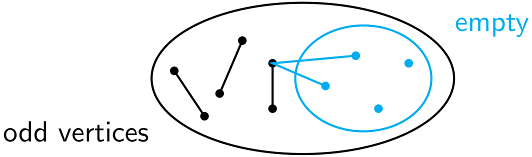

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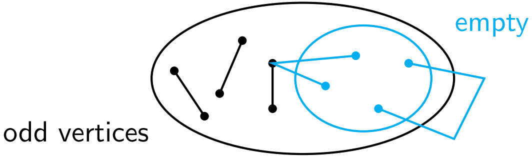

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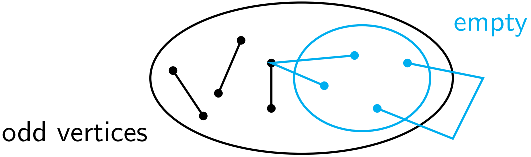

Two facts

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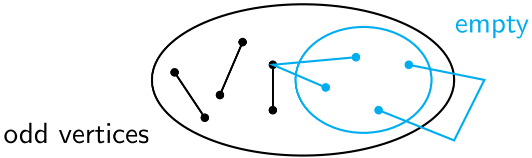

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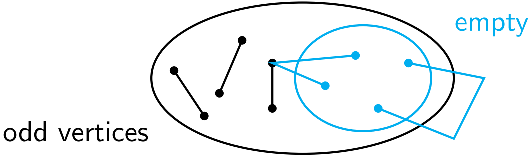

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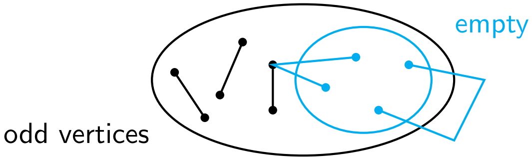

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For $p \gg \frac{\log n}{n}$ the product is $\frac{4\log n}{p} = o(n)$.

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Here:
We remove cycles of length $d \log^2 n$, so $\frac{n}{\log^2 n}$ cycles are enough to halve the average degree.

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If G has no cycle of length at least $3t$ then there is a set T of size at most t with $|N(T)| \leq 2|T|$.

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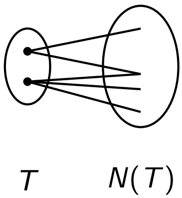

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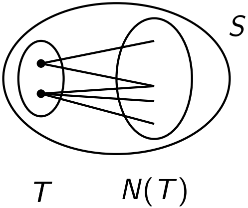

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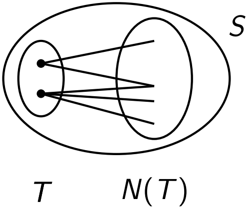

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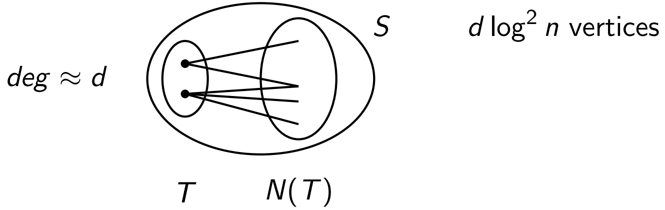

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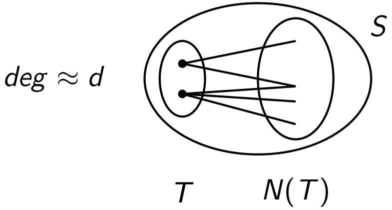

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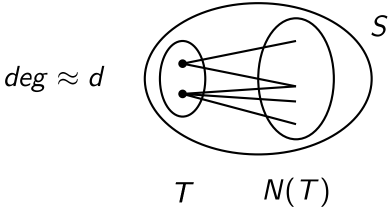

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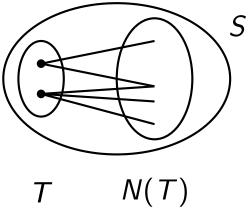

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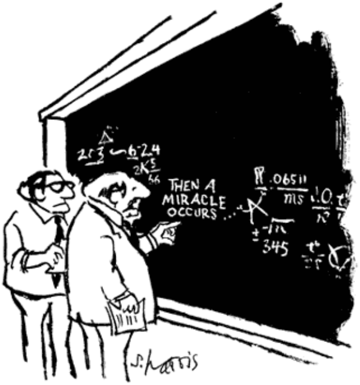

"I think you should be more explicit here in step two."

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Open problems

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   - How many edges need to be removed to make $G(n,p)$ Euler?
2. The Erdős–Gallai conjecture:

   Can any graph on n vertices be decomposed into $O(n)$ cycles and edges?

Thank you!