Large homogeneous submatrices

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EPFL

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joint work with János Pach and István Tomon

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Is there a linear-size ($\varepsilon n \times \varepsilon n$) homogeneous submatrix?

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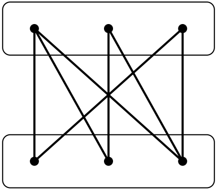

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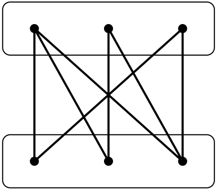

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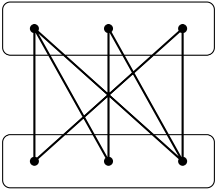

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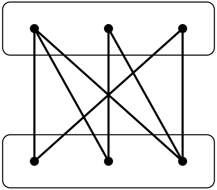

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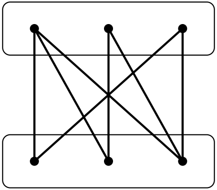

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$\Rightarrow$ There are linear-size subsets in both parts that together induce a complete or empty bipartite subgraph.

Points and directed lines

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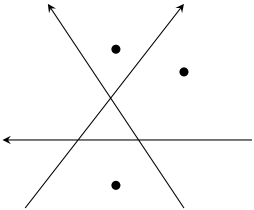

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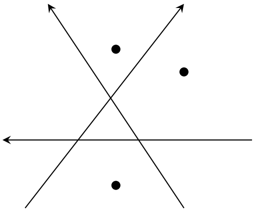

Matrix: $a_{ij} = \begin{cases} 1 \text{ if } i\text{'th point is on the right of } j\text{'th line} \\ 0 \text{ if } i\text{'th point is on the left of } j\text{'th line} \end{cases}$

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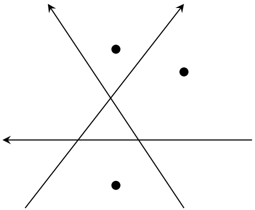

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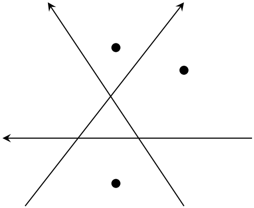

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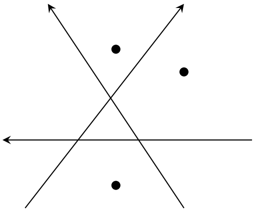

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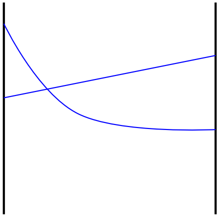

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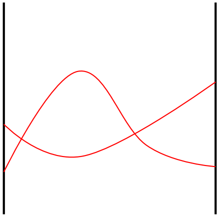

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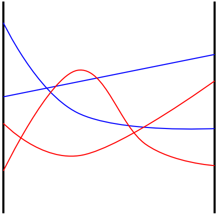

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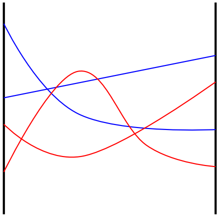

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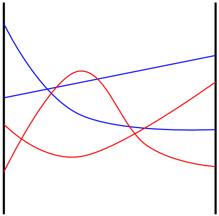

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Matrix is $\left(\begin{smallmatrix} 1 & 0 & 1 & 0 \cdots 1 & 0 \\ 0 & 1 & 0 & 1 \cdots 0 & 1 \end{smallmatrix}\right)$*-free* ($2k + 4$ *columns*).

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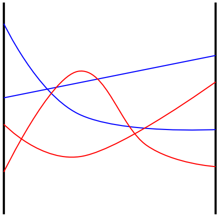

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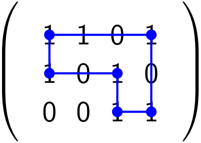

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If P is simple, then every $n \times n$ P-free 0-1 matrix has a linear-size homogeneous submatrix.

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All remaining simple matrices are 3×3, 3×4 and transposes.

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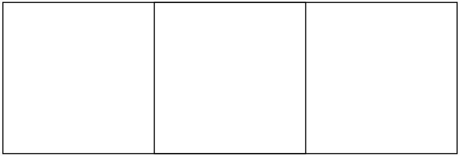

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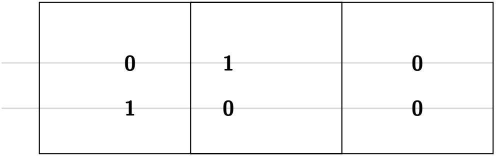

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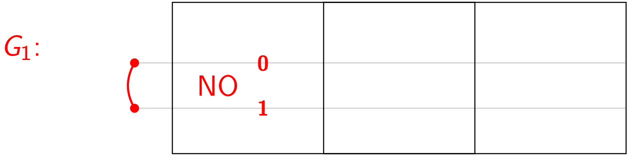

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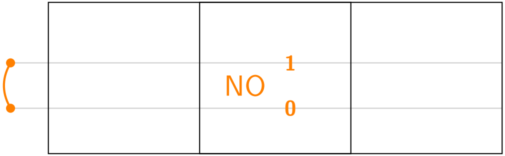

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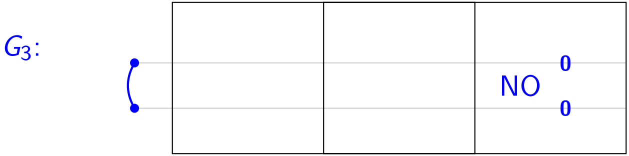

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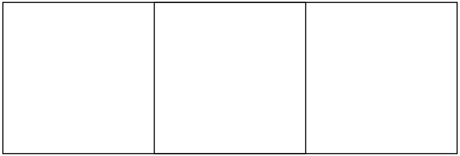

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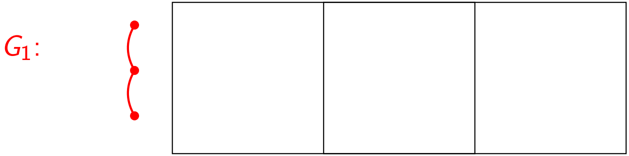

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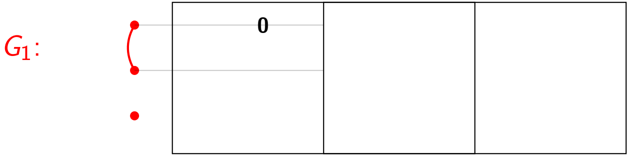

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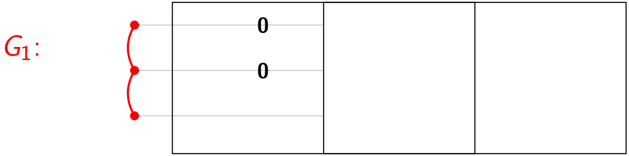

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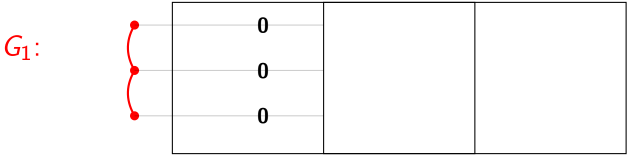

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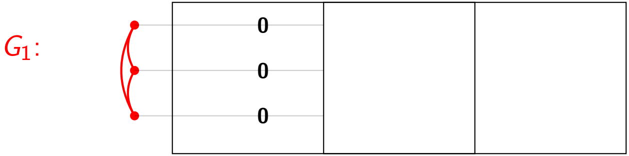

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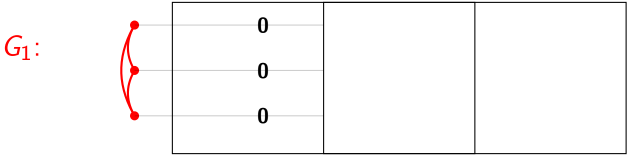

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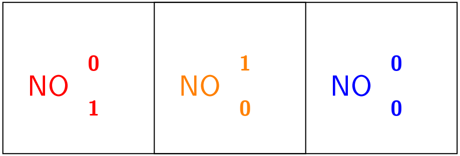

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Let G be the union of k comparability graphs on n vertices. Then G or $\overline{G}$ contains $K_{m,m}$ with $m \geq n^{1-o(1)}$.

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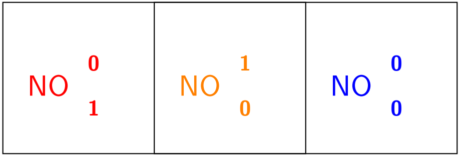

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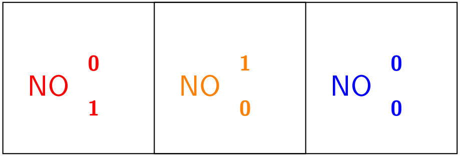

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- K-Tomon (2019): m cannot be replaced with $\frac{n}{\log^k n}$.

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If P is simple, then every $n \times n$ P-free 0-1 matrix has a linear-size homogeneous submatrix.

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If P is acyclic, then every $n \times n$ P-free 0-1 matrix with $\Theta(n^2)$ 0-entries has a linear-size all-0 submatrix.

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Thank you!