Covering random graphs by monochromatic cycles

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joint work with
F. Mousset, R. Nenadov, N. Škorić and B. Sudakov

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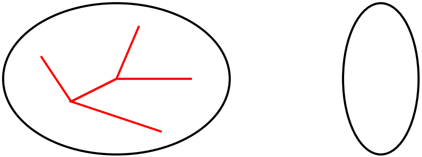

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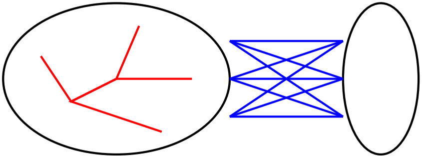

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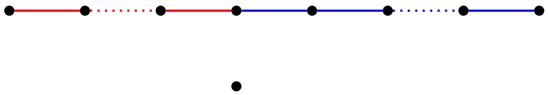

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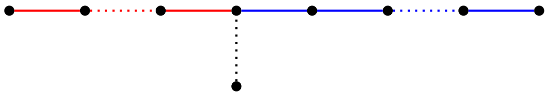

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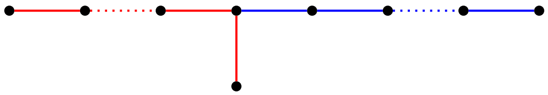

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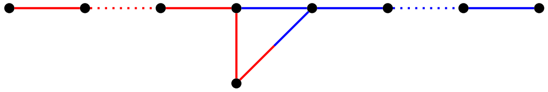

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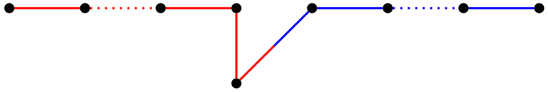

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Our results are about covering the Erdős-Rényi random graph $G_{n,p}$ with monochromatic cycles.

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- Bal–DeBiasio: 2-colored case when $p \gg (\frac{\log n}{n})^{1/3}$.
- Kohayakawa–Mota–Schacht [2016+]: conjecture holds for $r = 2$.

Our result

Theorem (K–Mousset–Nenadov–Škorić–Sudakov, 2017+)

If $p > n^{-1/r+\varepsilon}$ *then the vertices of an* r*-edge-colored* $G_{n,p}$ *can be covered by* $O(r^8 \log r)$ *monochromatic cycles whp.*

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The proof heavily exploits covering instead of partitioning.

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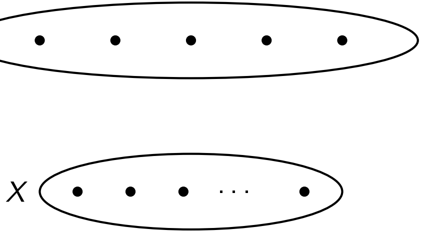

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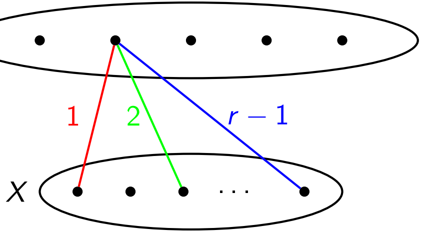

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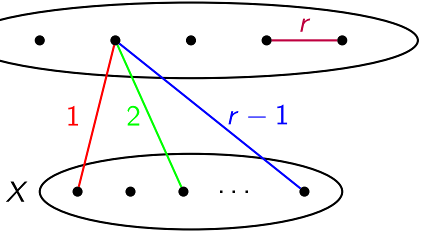

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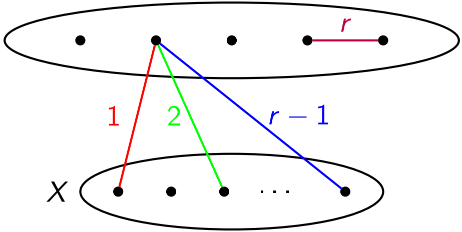

No monochromatic path between two vertices of X.

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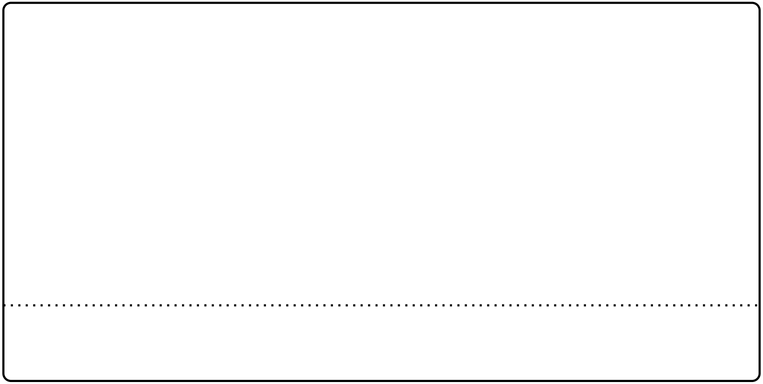

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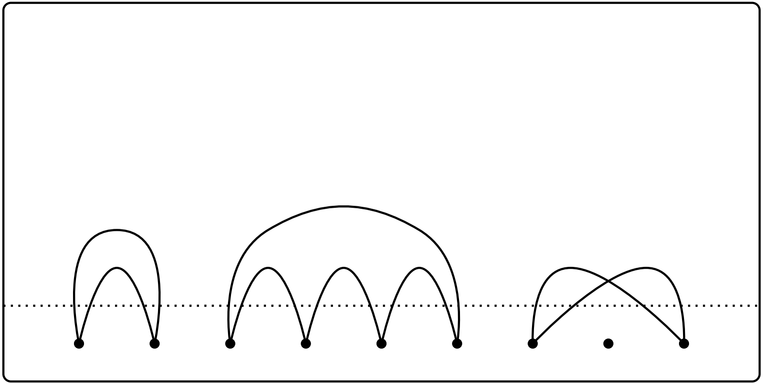

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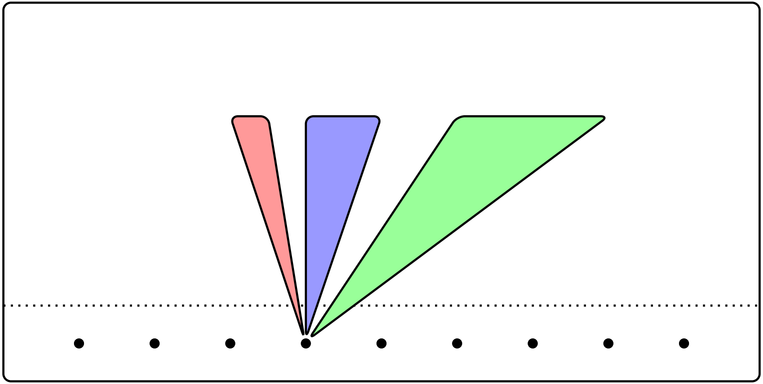

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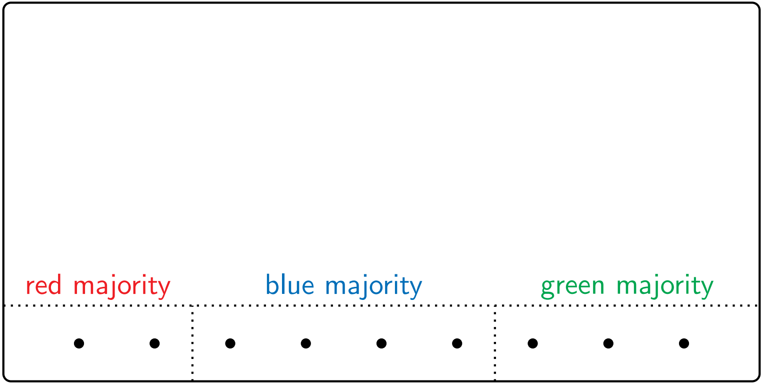

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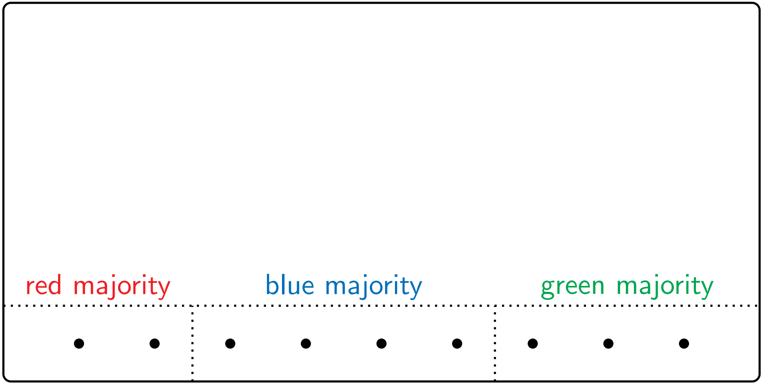

We handle each color separately.

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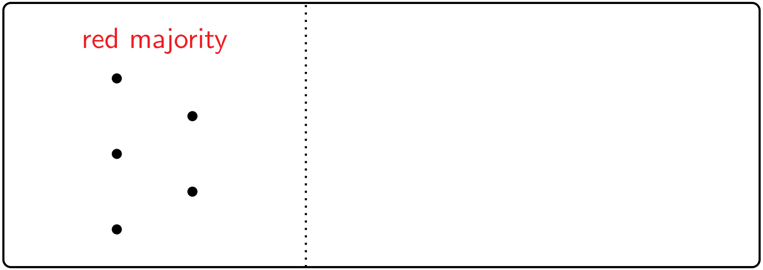

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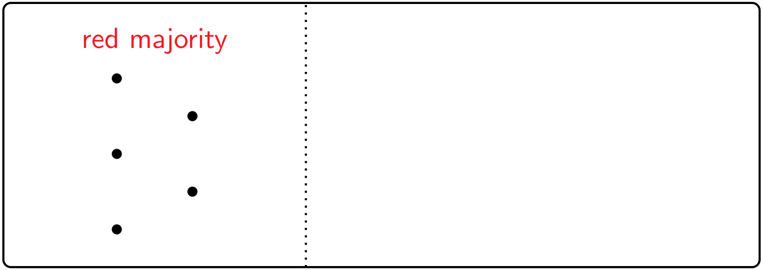

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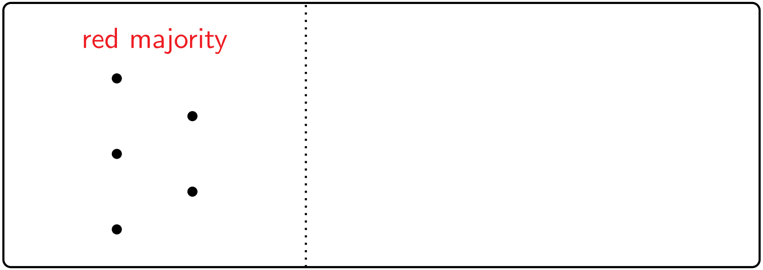

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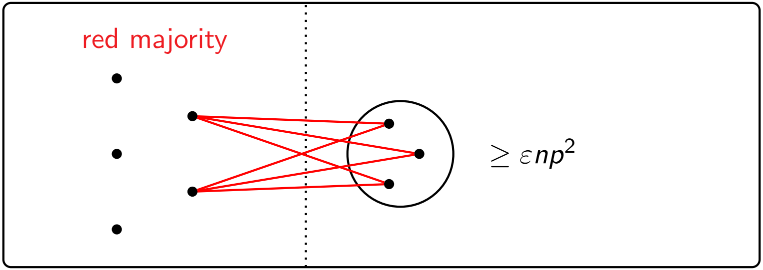

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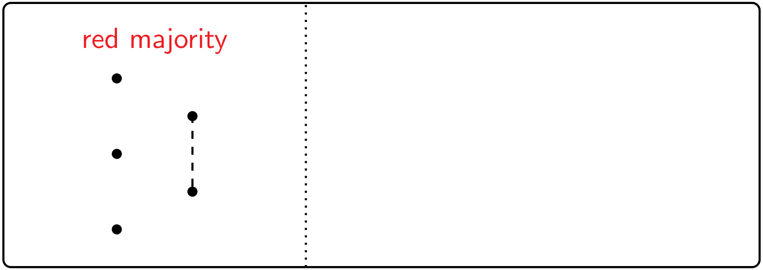

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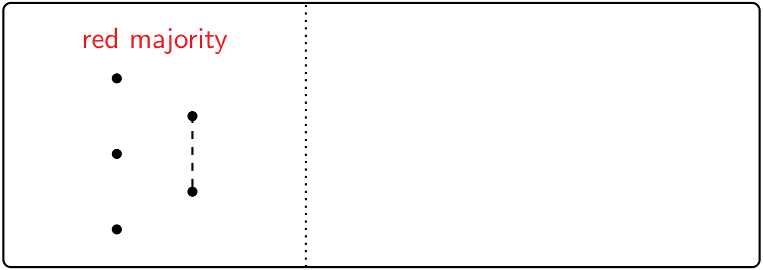

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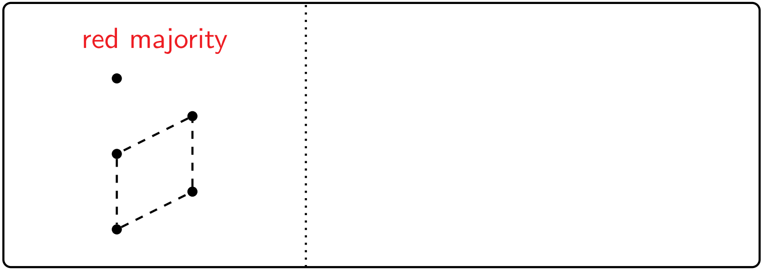

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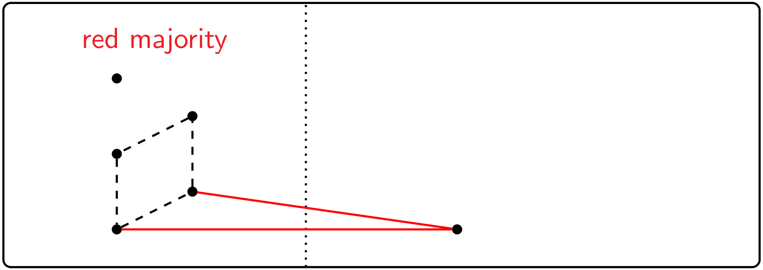

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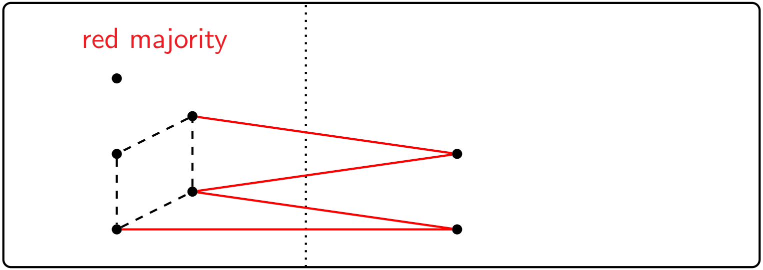

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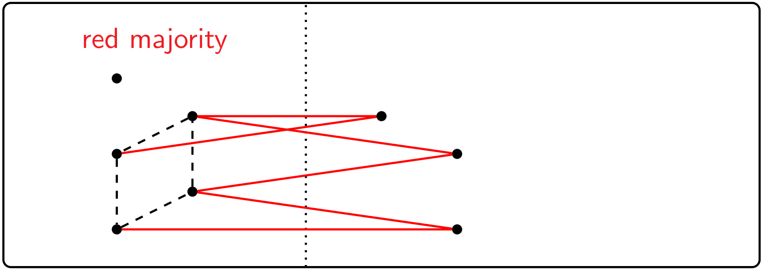

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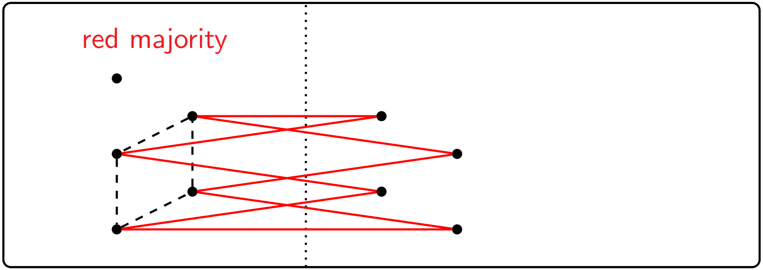

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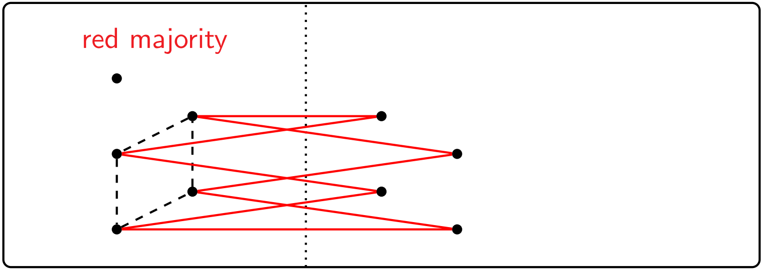

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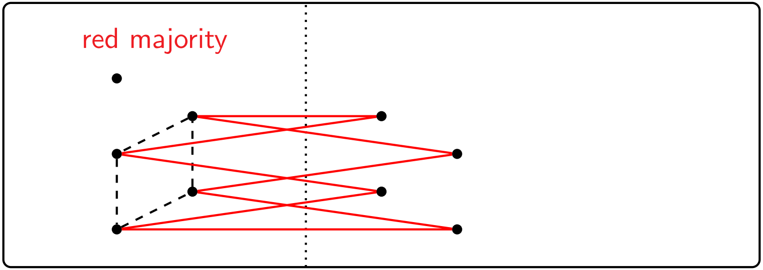

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Goal: cover auxiliary graph using few cycles.

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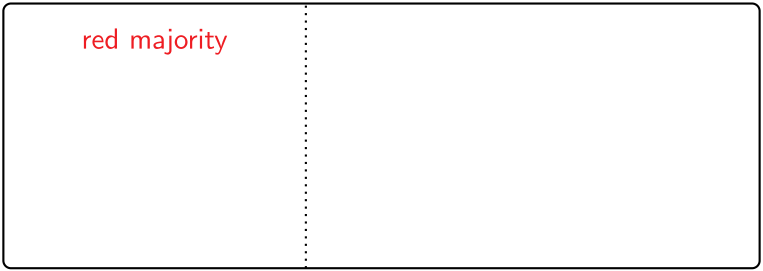

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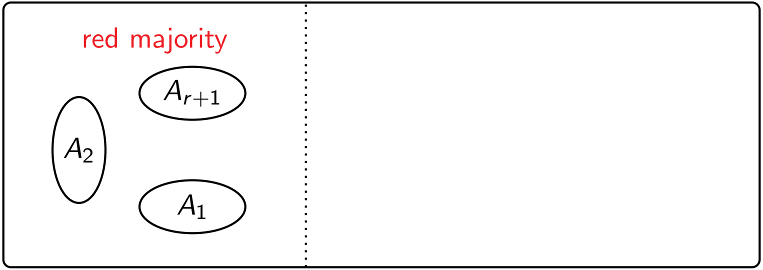

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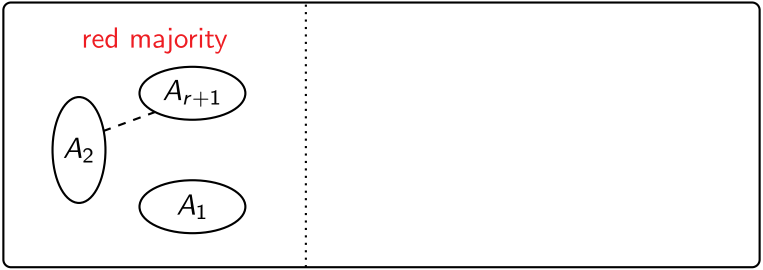

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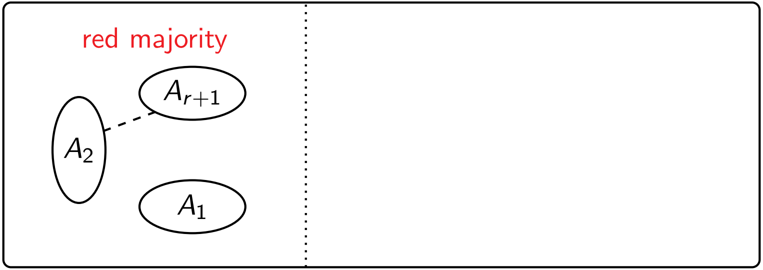

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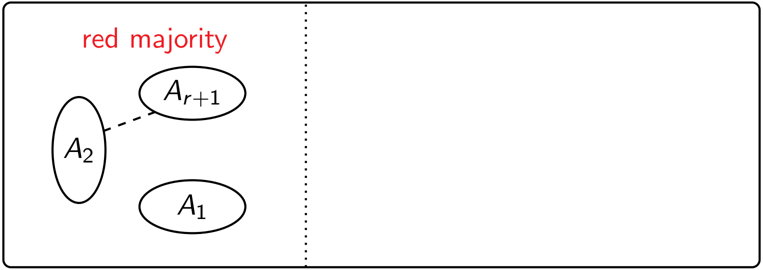
Lemma (KMNŠS)

If the complement of G does not contain a $K_{m,\ldots,m}$ (with r parts), then G has $r-1$ (disjoint) cycles covering all but $O(r^4 m)$ vertices.

Covering all but $O(1/p)$ vertices



Let $A_1, \ldots, A_{r+1}$ be sets of size $1/p$.

Claim

There is an auxiliary edge between some A_i and A_j

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Generalizes a lemma by Ben-Eliezer–Krivelevich–Sudakov.

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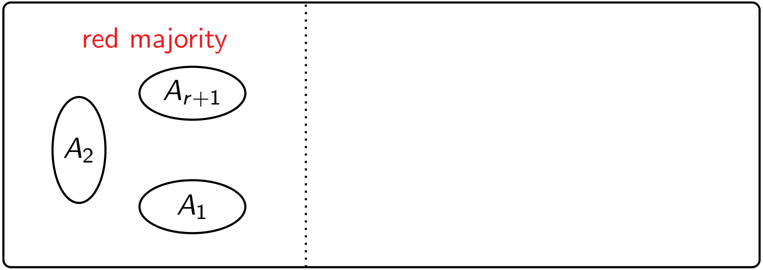

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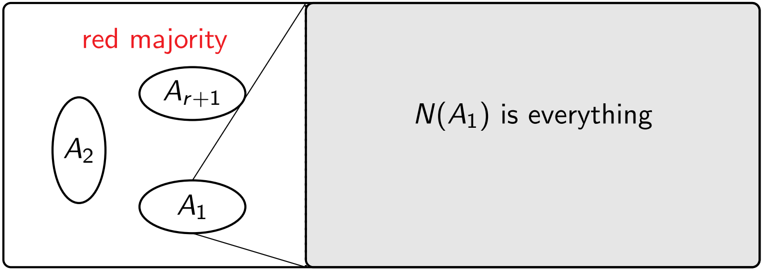

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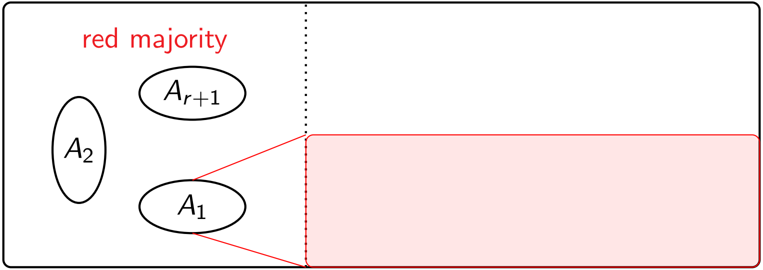

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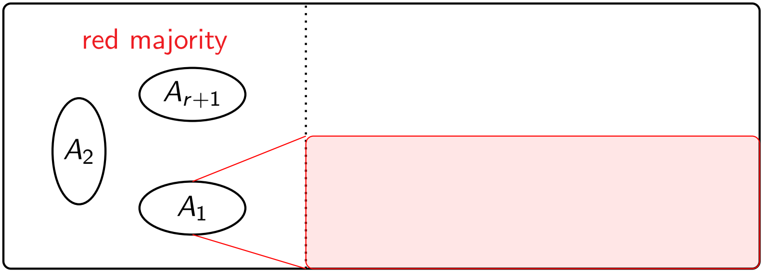

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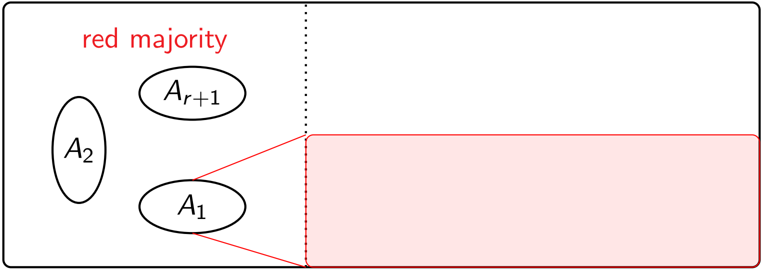

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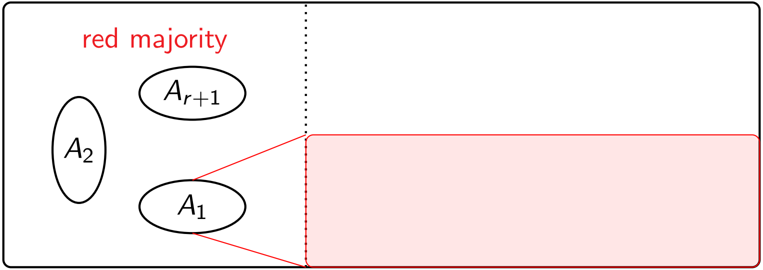

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- Some vertices in A_i, A_j will have εnp^2 common red neighbors

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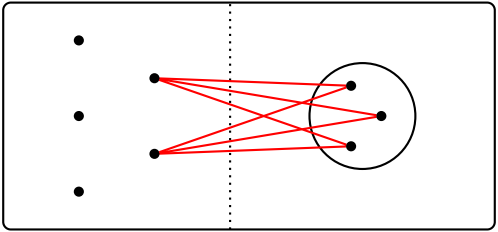

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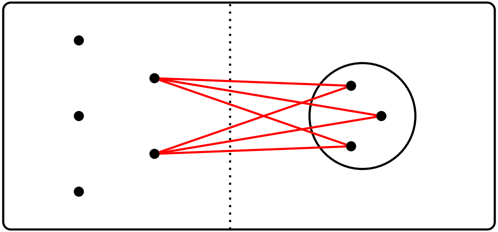

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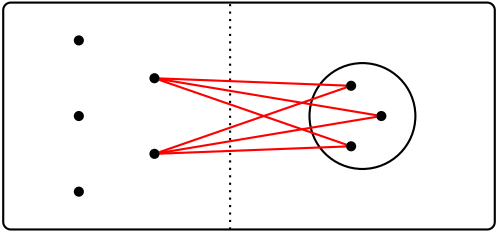

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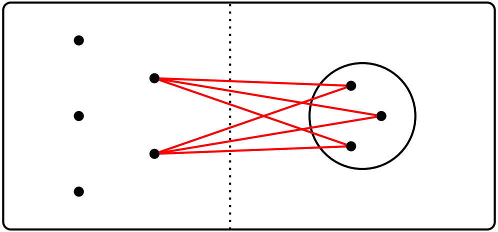

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To push it down to $p > n^{-1/r+\varepsilon}$, we consider longer paths.

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Thank you!