On the Turán number of ordered forests

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joint work with Gábor Tardos, István Tomon and Craig Weidert

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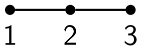

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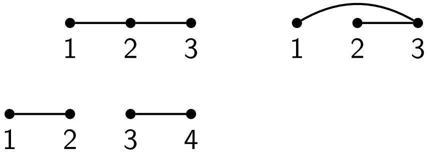

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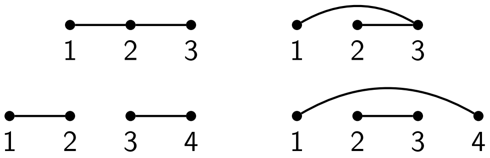

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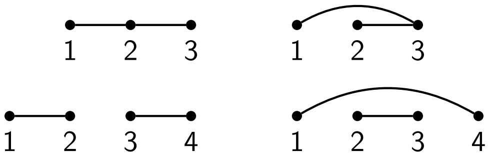

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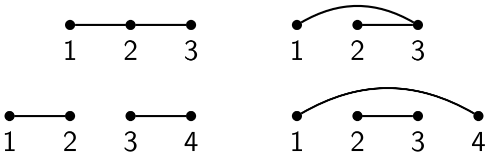

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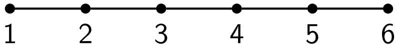

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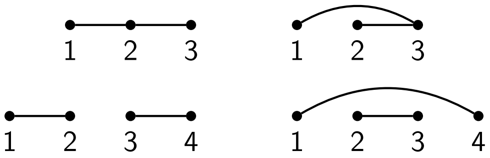

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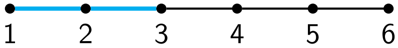

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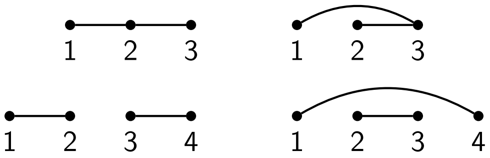

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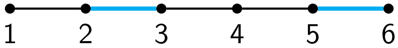

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The Ramsey problem for ordered graphs was recently studied by Balko–Cibulka–Král–Kynčl and Conlon–Fox–Lee–Sudakov.

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Theorem (K–Tardos–Tomon–Weidert)

Yes, for a large class of ordered forests.

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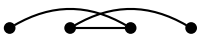
- $\mathrm{ex}_<(n,$  $) = \Theta(n \log n)$.

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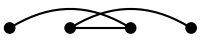
- $\mathrm{ex}_<(n,$  $) = \Theta(n \log n)$.

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Our result proves $\mathrm{ex}_<(n, H) = n^{1+o(1)}$ for a large class of forests.
(This class covers all previously known forests, and more.)

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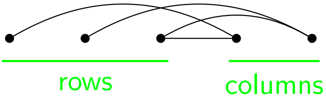

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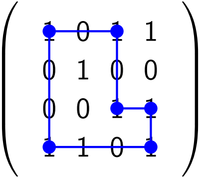

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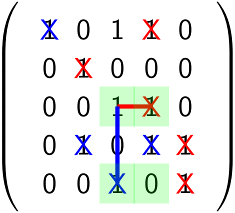

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$$
\begin{pmatrix}
1 & & 1 & 1 & \\
 & 1 & & & 1 \\
1 & & 1 & 1 & \\
 & 1 & & 1 & 1 \\
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\end{pmatrix}
$$

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$$\begin{pmatrix} 1 & & 1 & \cancel{1} & \\ & 1 & & & \cancel{1} \\ 1 & & 1 & \cancel{1} & \\ & 1 & & 1 & \cancel{1} \\ 1 & & 1 & & \cancel{1} \end{pmatrix}$$

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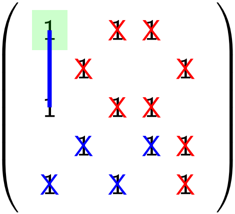

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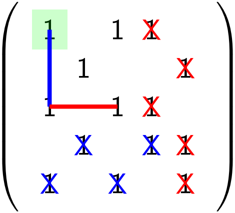

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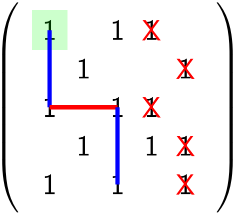

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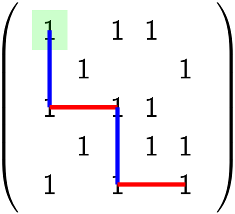

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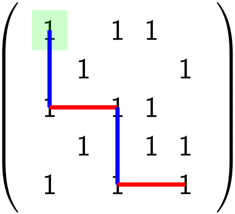

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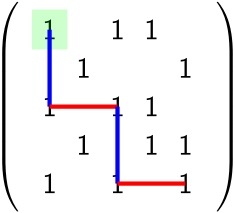

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- Marcus–Tardos (2004): $\mathrm{ex}(n, P) = O(n)$ if P is permutation

But: $\mathrm{ex}(n,($$)) = \Omega(n \log n)$

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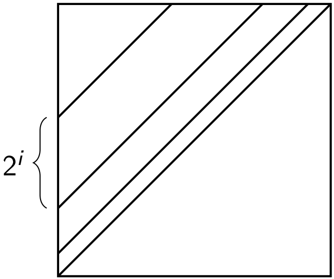

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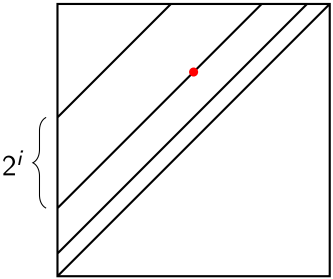

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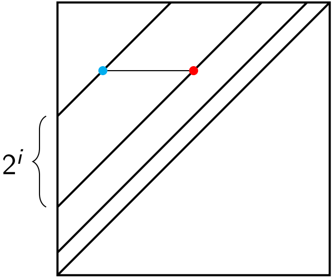

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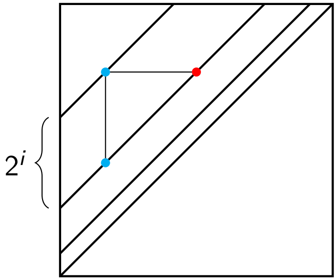

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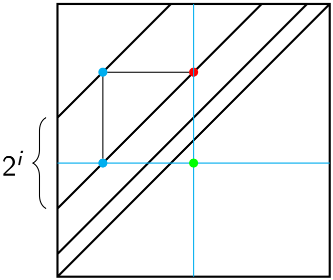

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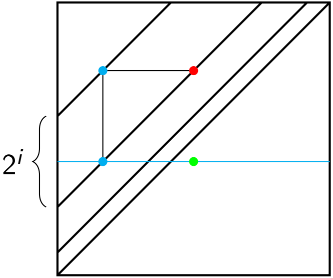

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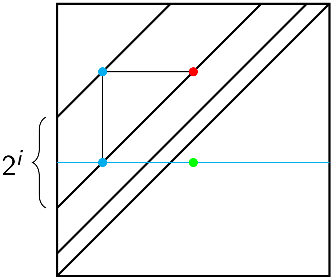

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Is $\mathrm{ex}(n, P) = O(n \log n)$ for every acyclic P?

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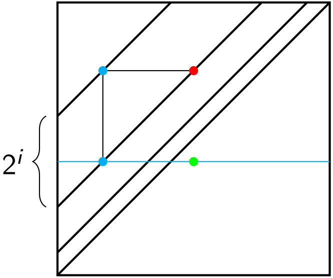

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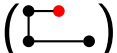
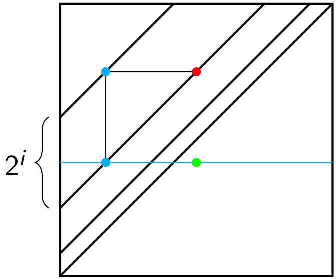
But: $\mathrm{ex}(n,($  $)) = \Omega(n \log n)$



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Conjecture (Pach–Tardos, 2006)

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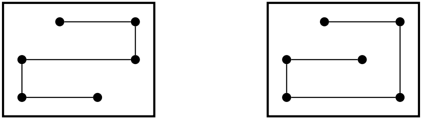

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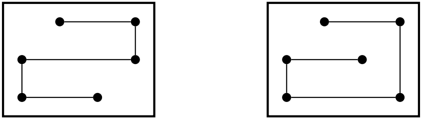

Theorem (K–Tardos–Tomon–Weidert, 2017+)

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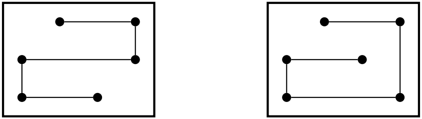

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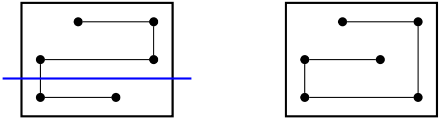

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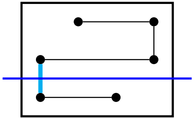

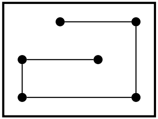

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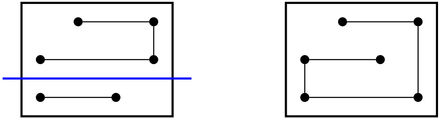

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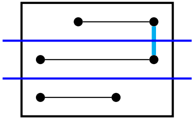

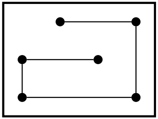

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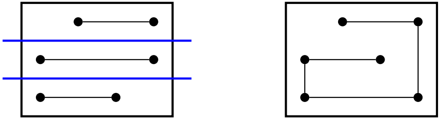

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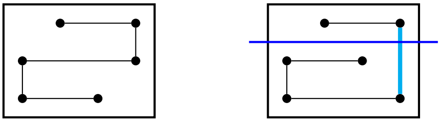

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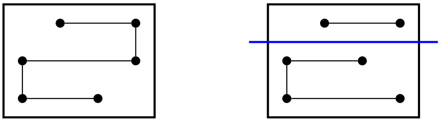

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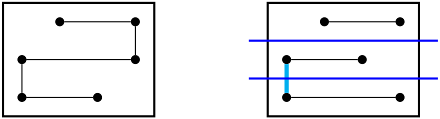

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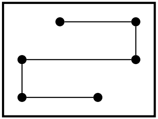

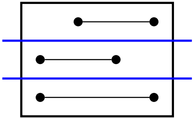

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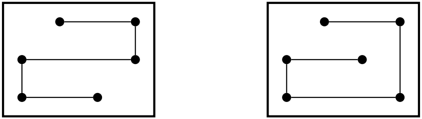

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- Includes all patterns with 3 rows.

Embedding strategy

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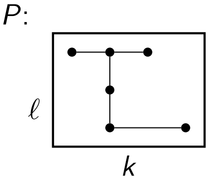

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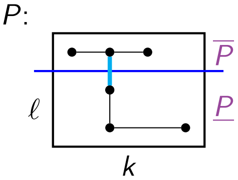

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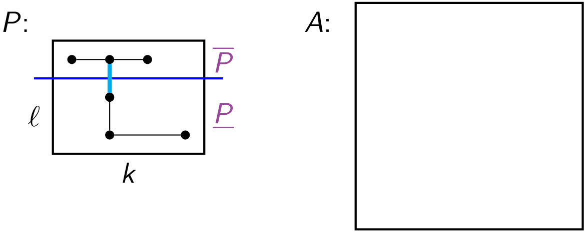

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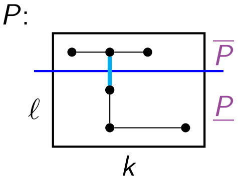

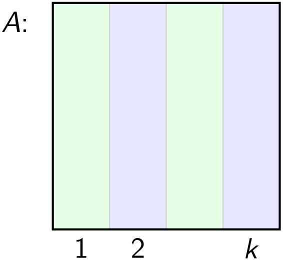

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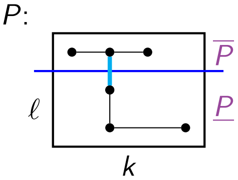

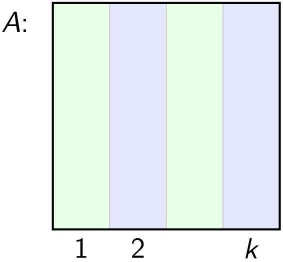

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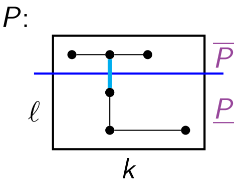

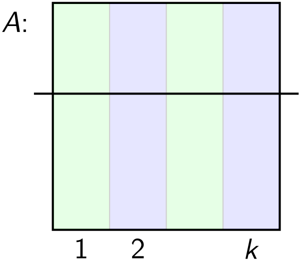

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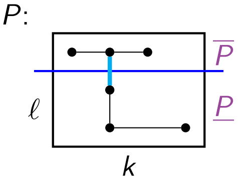

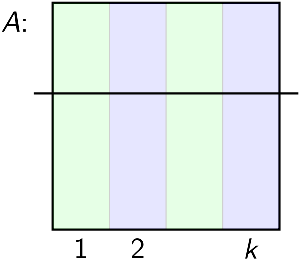

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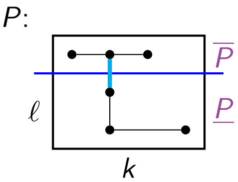

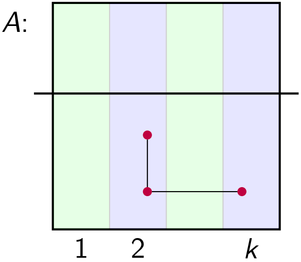

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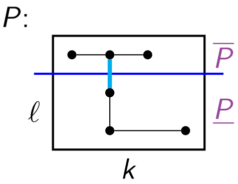

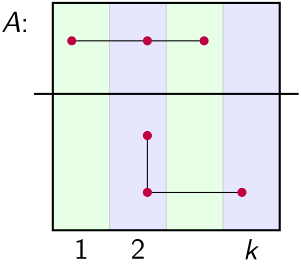

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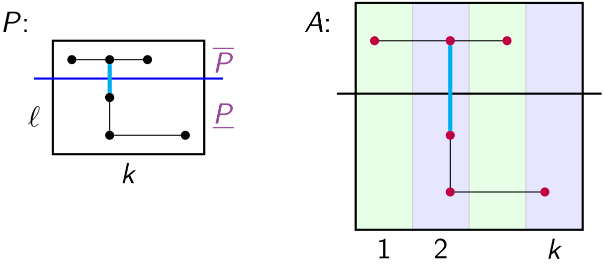

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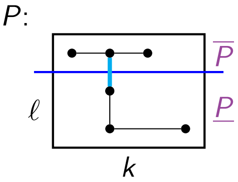

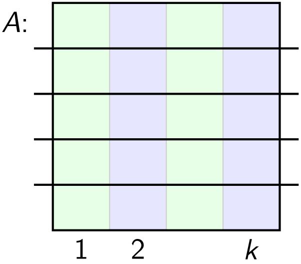

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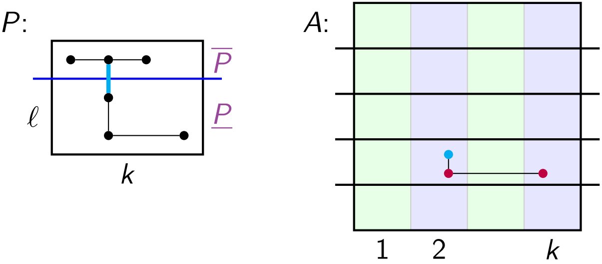

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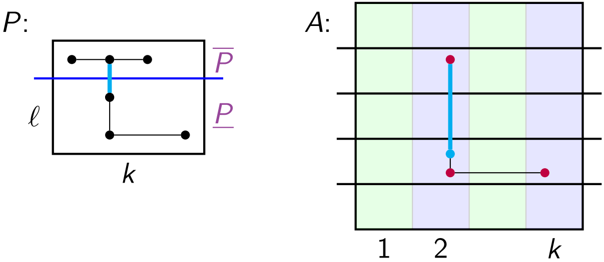

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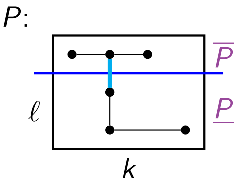

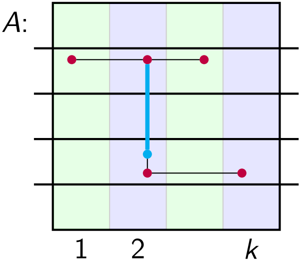

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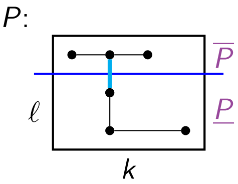

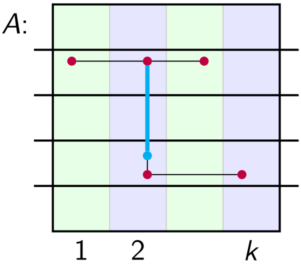

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Claim

If A has at least $n^{1+\varepsilon}$ 1's then **either** the above embedding works, **or** we can find a denser submatrix A' with at least $n'^{1+\varepsilon+\varepsilon^{\ell}}$ 1's.

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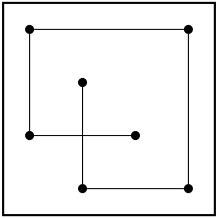

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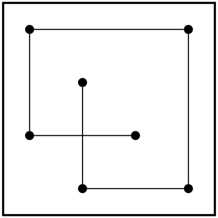

Thank you!