Improved Ramsey-type results in comparability graphs

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EPFL

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joint work with István Tomon

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"Dilworth":
If X has no t-chain, then it can be partitioned into t antichains.

Convex sets in the plane

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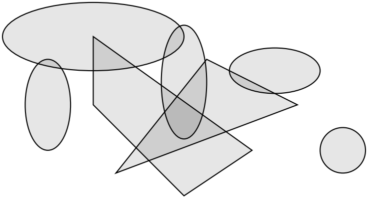

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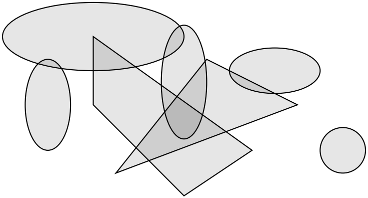

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Given n convex sets in the plane, how big is the largest disjoint or pairwise intersecting subfamily?

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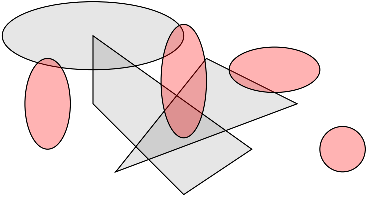

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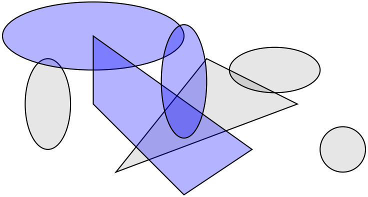

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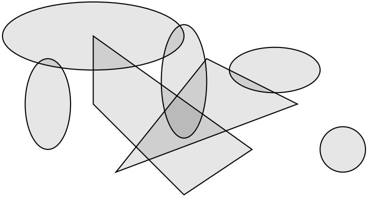

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Theorem (Larman-Matoušek-Pach-Törőcsik, 1994)

At least $n^{1/5}$.

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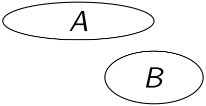
A, B disjoint, B is “below” A



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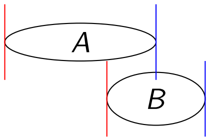

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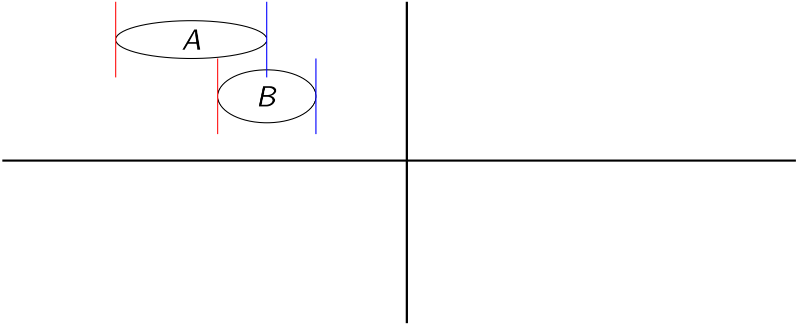

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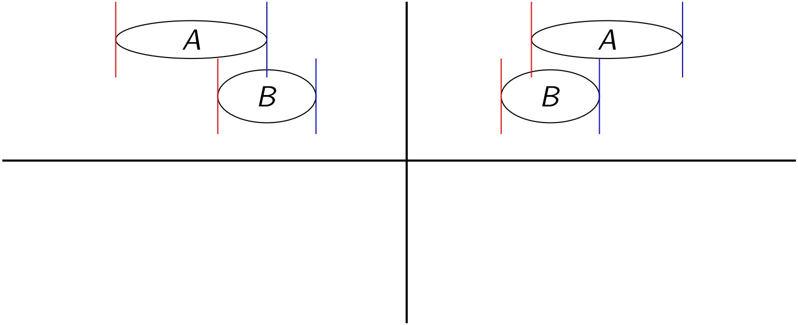

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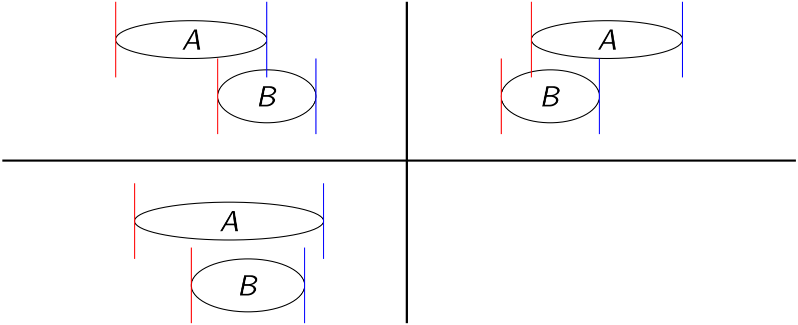

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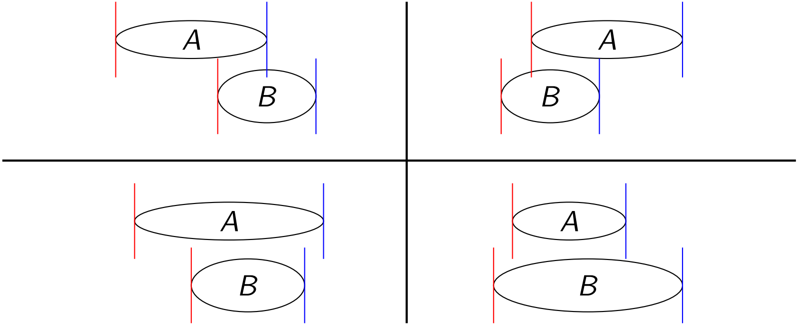

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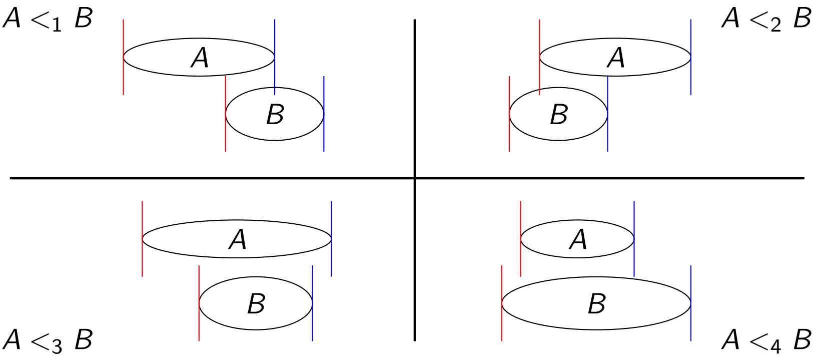

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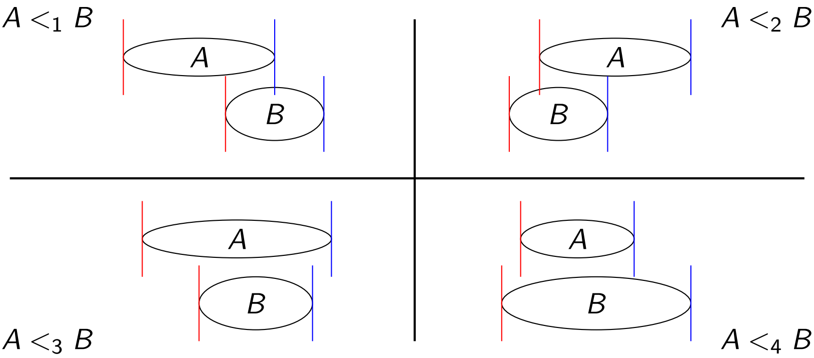

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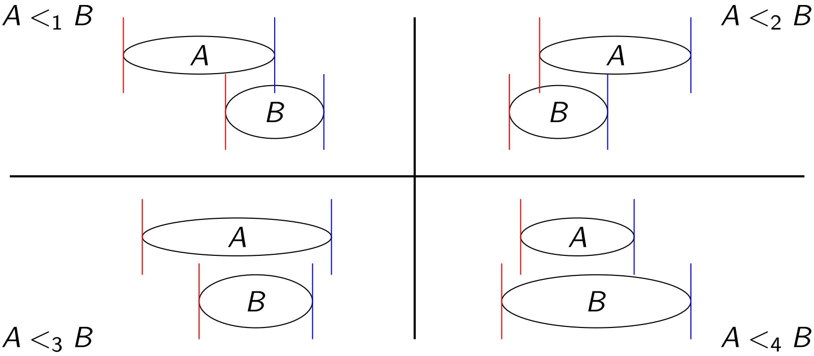

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$\Rightarrow$ S_4 incomparable in all 4 posets $\rightarrow$ intersecting family.

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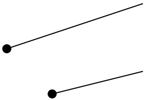

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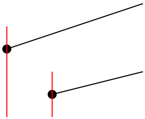

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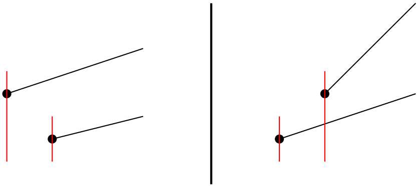

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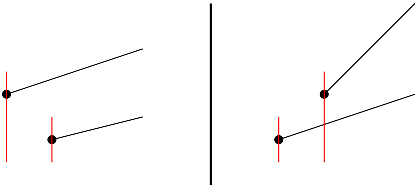

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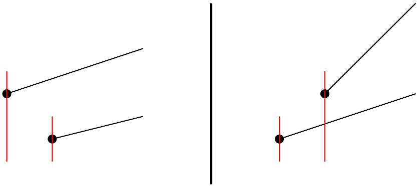

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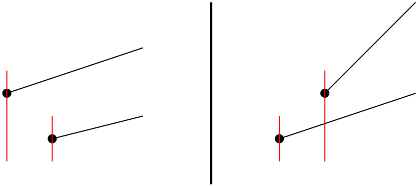
	Lower bound	Upper bound
convex sets	$n^{1/5}$	$n^{0.405}$ (Kynčl)
halflines	$n^{1/3}$	$n^{0.431}$ (LMPT)



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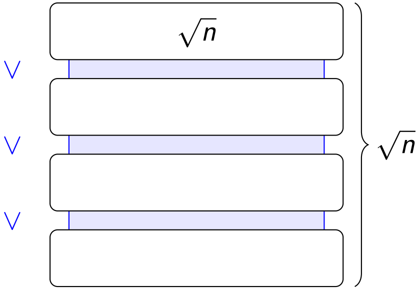

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Theorem (K-Tomon, 2019+)

$f_2(n) = n^{1/3+o(1)}$

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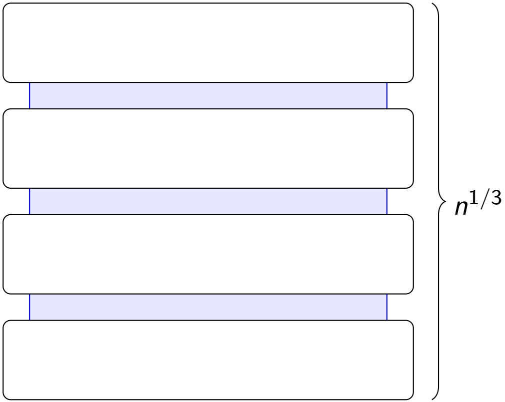

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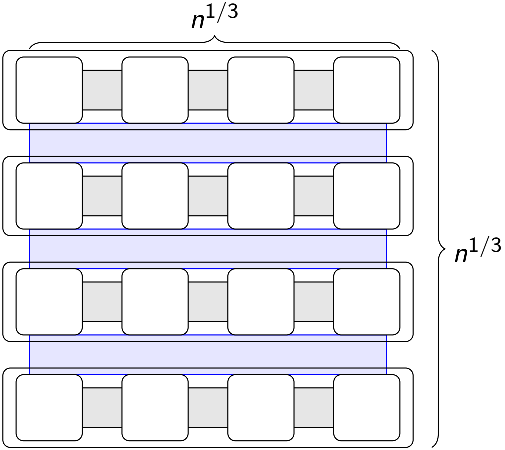

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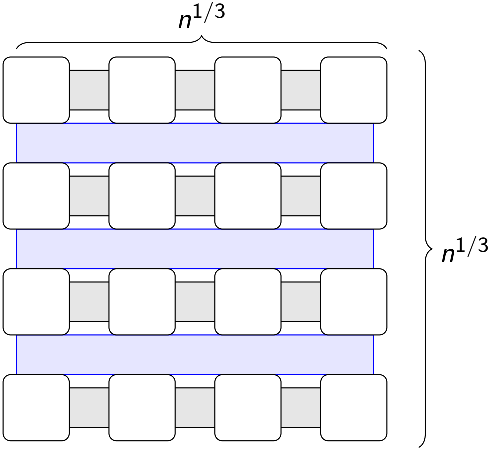

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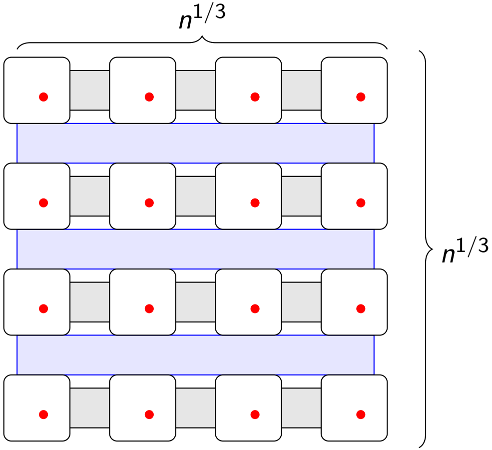

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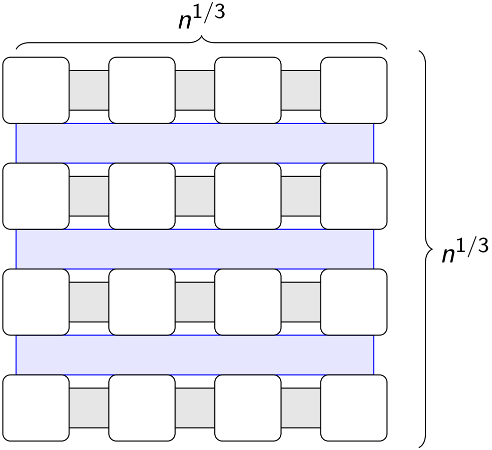

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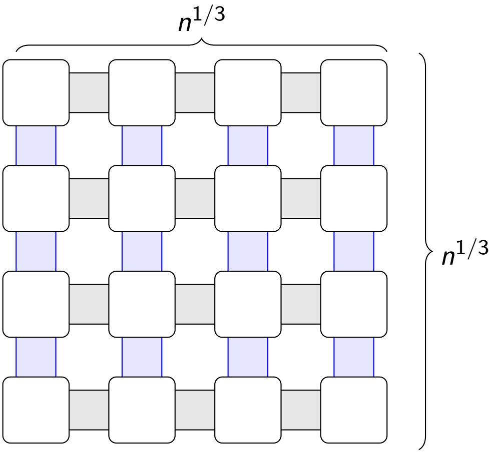

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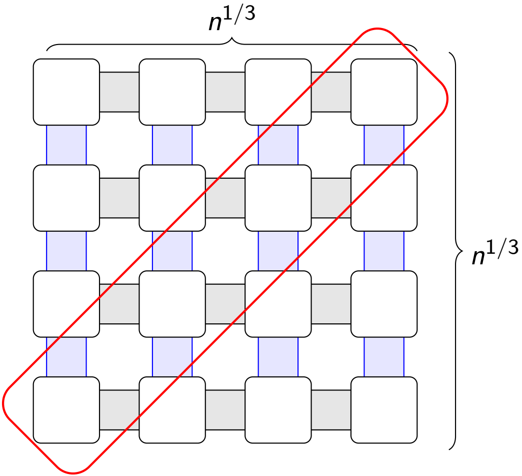

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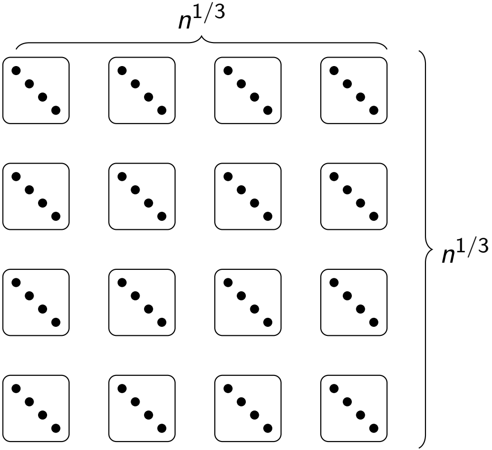

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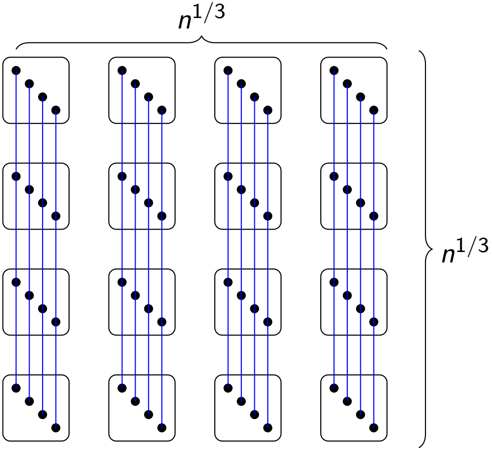

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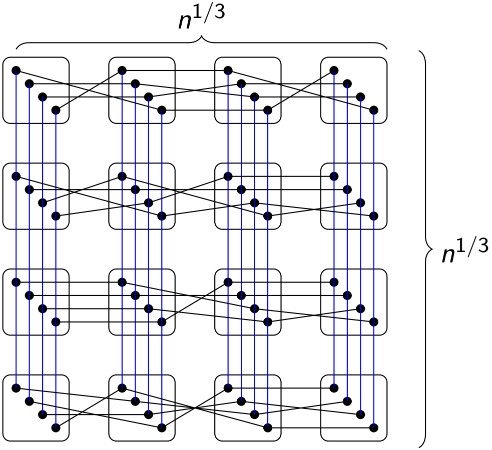

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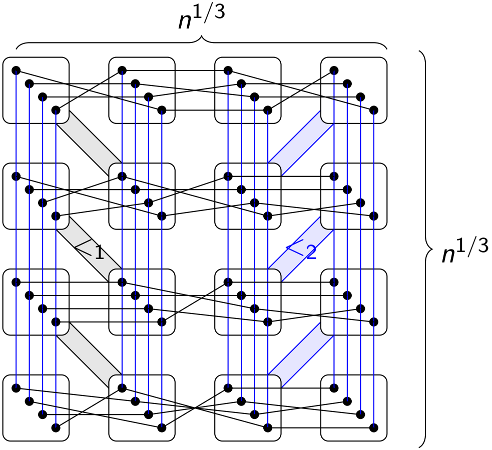

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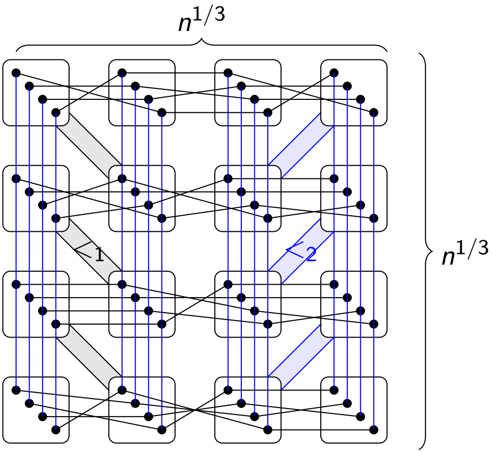
Claim 1:

$\alpha(G) = n^{1/3}$



Construction for $f_2(n) \leq n^{1/3+o(1)}$

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Claim 2:
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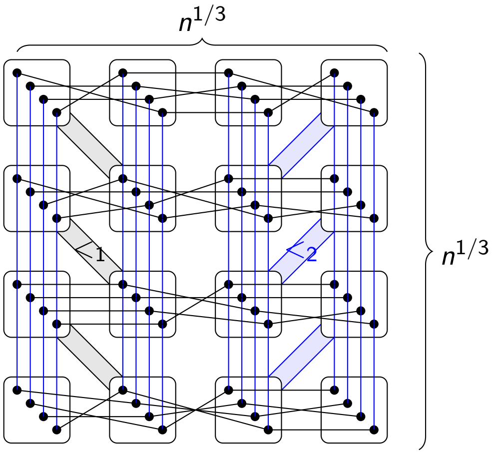

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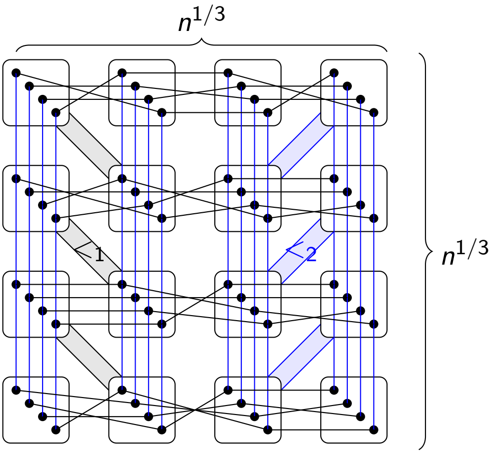

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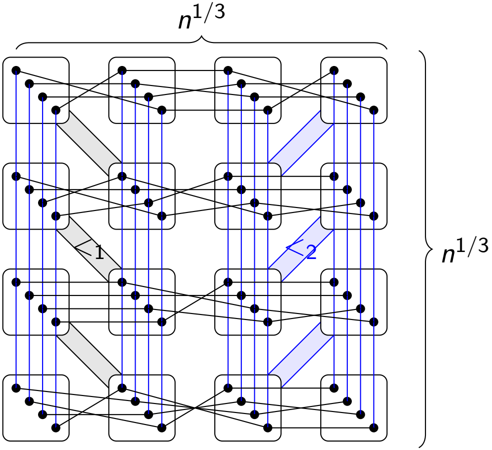

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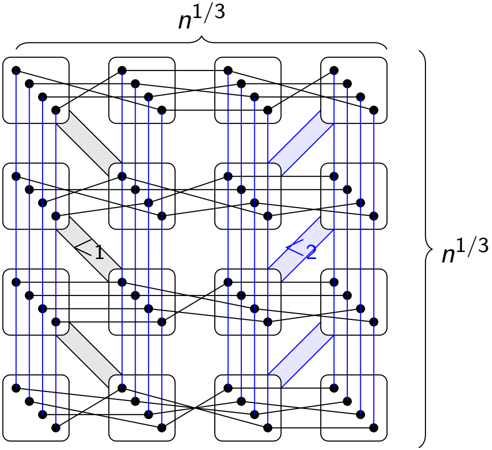

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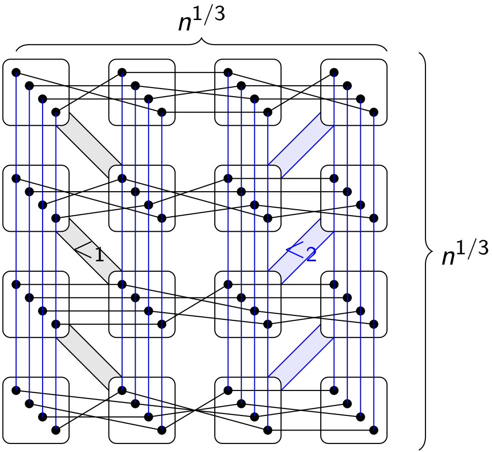

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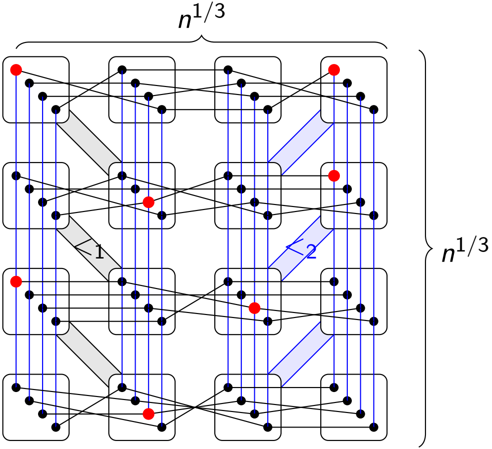

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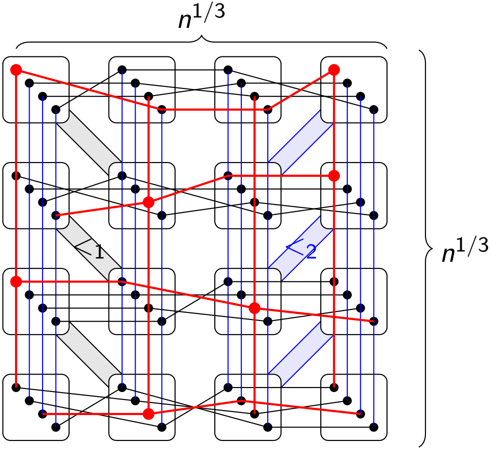

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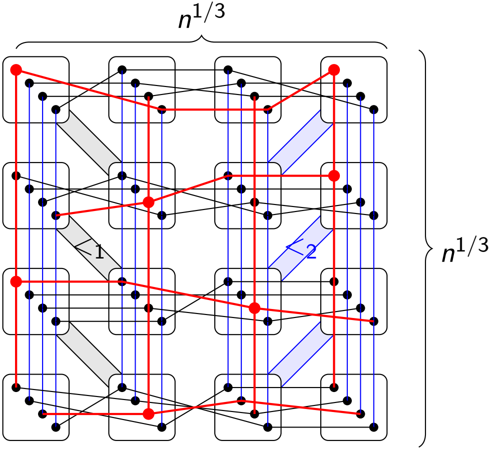

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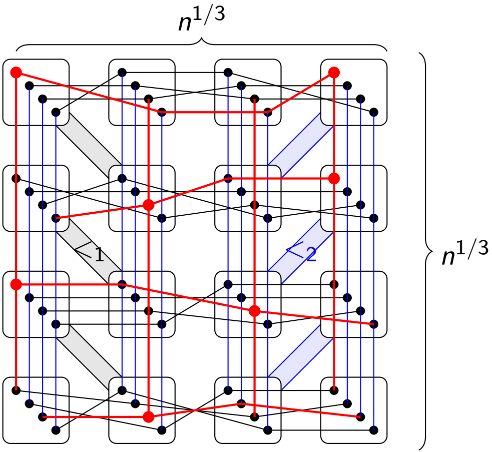

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- $\mathbf{P}$(fixed row chain intersects fixed column chain) $= n^{-1/3}$

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- Improving $n^{1/3}$ for halflines needs different approach

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Theorem (K-Tomon, 2019+)

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Thank you!