Saturation in random graphs

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joint work with Benny Sudakov

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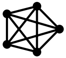

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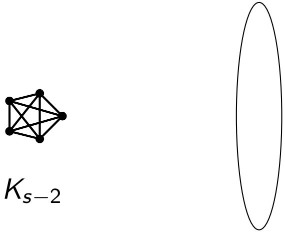

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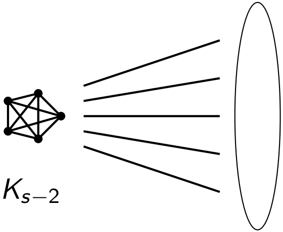

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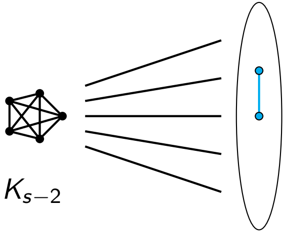

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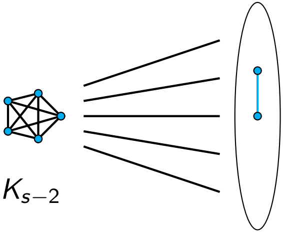

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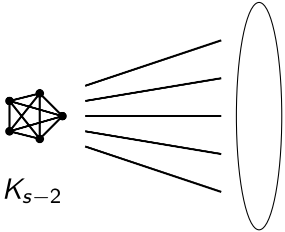

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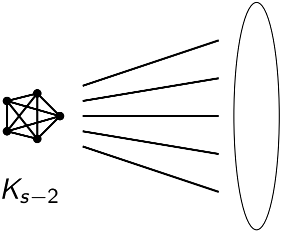

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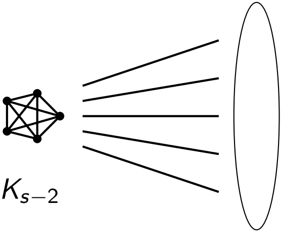

$$\binom{n}{2} - \binom{n-s+2}{2} \text{ edges}$$

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Theorem (Zykov, 1949; Erdős–Hajnal–Moon, 1964)

A K_s-saturated n-vertex graph has at least $\binom{n}{2} - \binom{n-s+2}{2}$ edges.

Saturation in different host graphs

Bipartite saturation problem (Erdős–Hajnal–Moon, 1964)

What is the minimum number of edges a $K_{s,s}$-saturated n-by-n bipartite graph can have?

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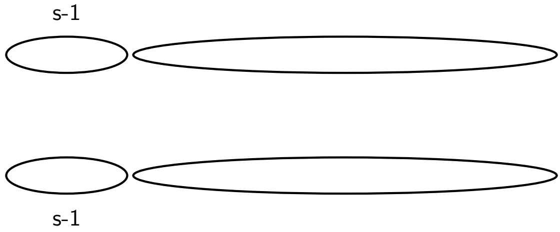

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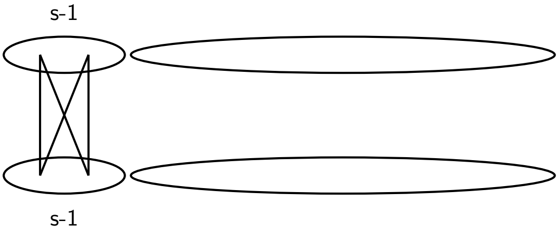

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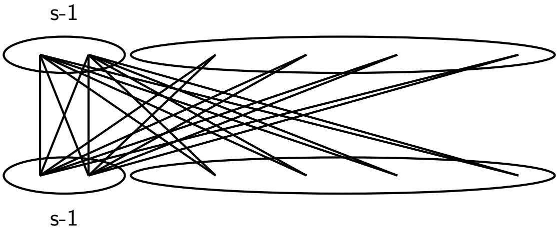

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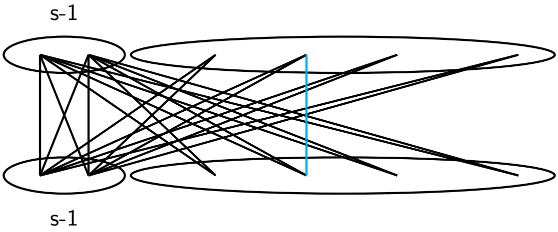

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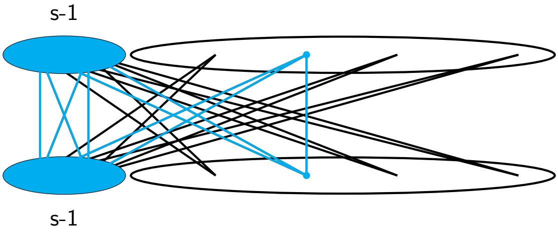

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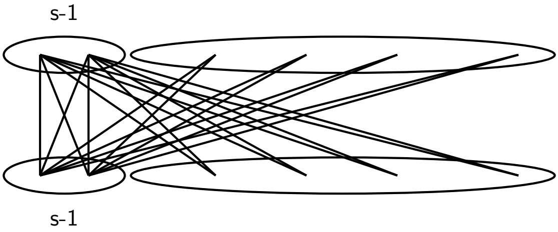

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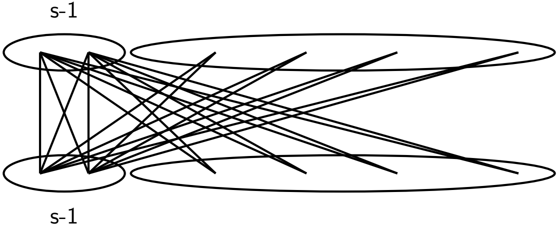

Theorem (Wessel, 1966; Bollobás, 1967)

The above construction is optimal.

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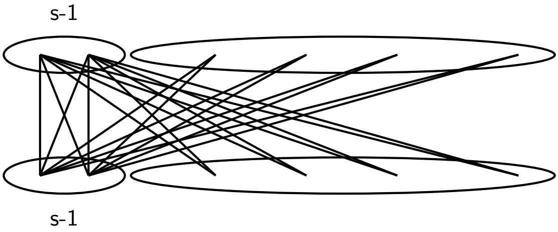

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Saturation has also been studied in e.g. hypercubes and complete multipartite graphs.

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Question

What is the minimum number of edges a K_s-saturated subgraph in $G(n, p)$ graph can have?

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can also be proved using hypergraph containers (Balogh–Morris–Samotij, Saxton–Thomason)

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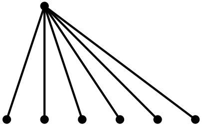

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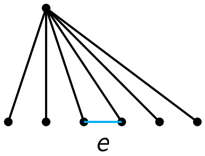

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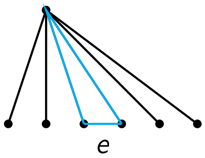

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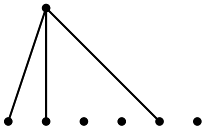

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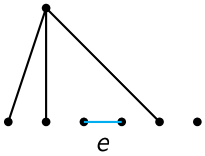

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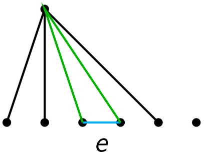

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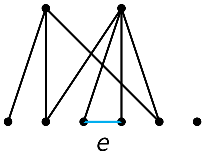

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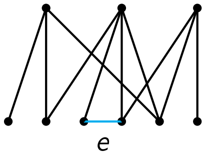

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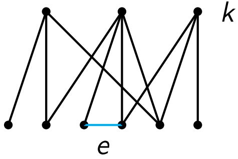

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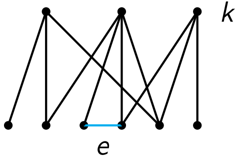

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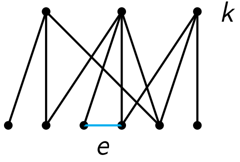

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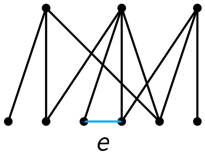

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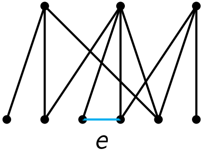

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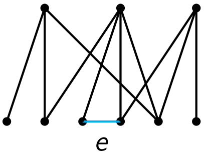

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Aim : The minimum is $(1 + o(1))\frac{n}{2}k \approx n \log_{4/3} n$.

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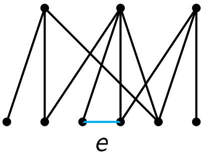

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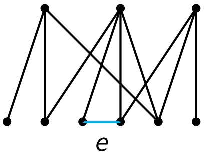

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Theorem (K–Sudakov)

The minimum is $(1 + o(1))n \log_2 n$.

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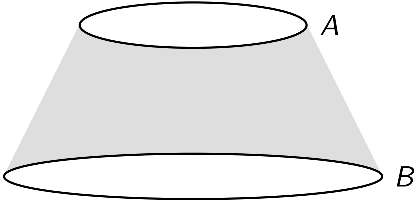

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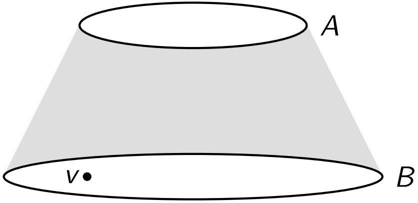

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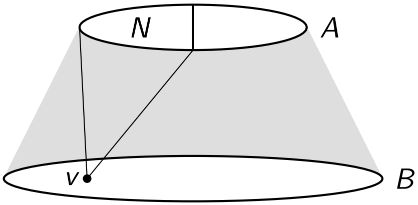

If v has more than $\log_2 n$ neighbors in A then all edges at v are covered.

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- Any set of size $(1+\varepsilon)\log_2 n$ is adjacent to all other vertices whp.
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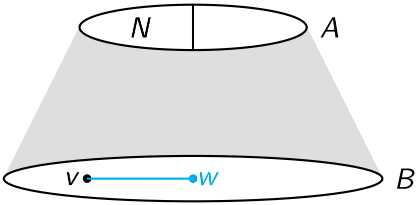

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Why $\log_2 n$?

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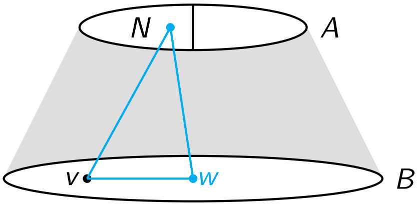

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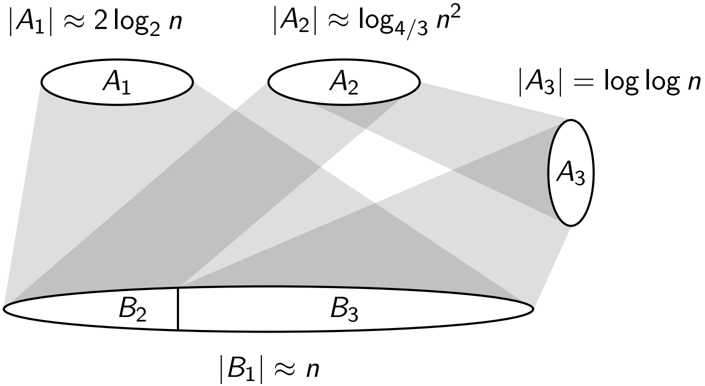
The upper bound construction



- $B_2 =$ vertices with at most $\log_2 n$ neighbors in A_1.
  $|B_2| = o(n)$.

The lower bound

Claim

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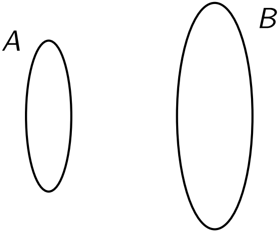

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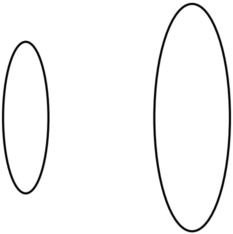
If H is a triangle-saturated subgraph of $G = G(n, 1/2)$ then it has at least $(1 + o(1))n \log_2 n$ edges.

$\{u : d_H \geq \log^2 n\} = A$



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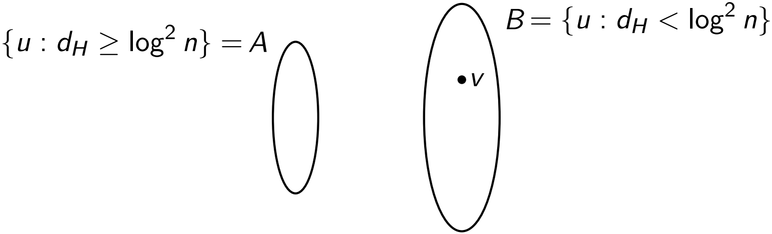

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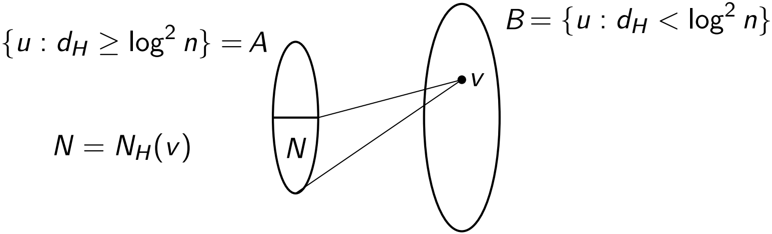

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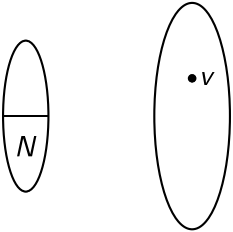

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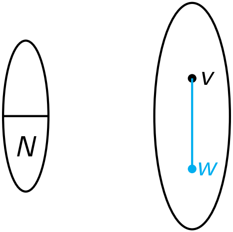

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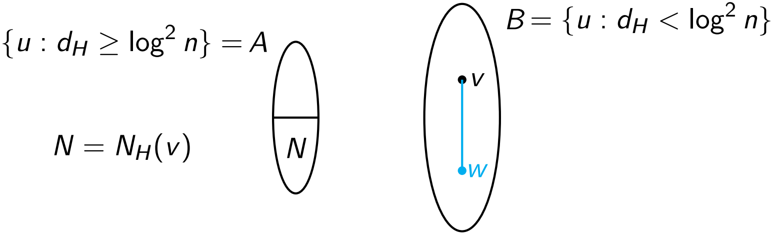

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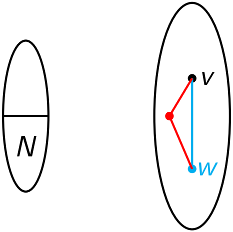

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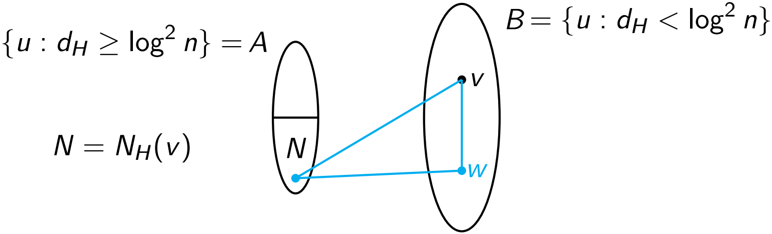

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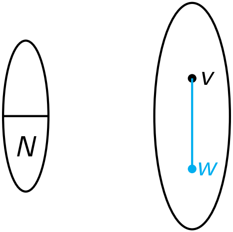

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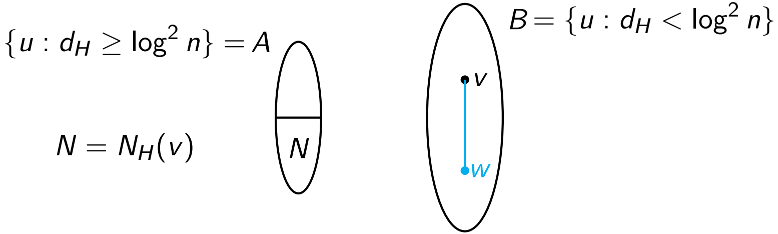

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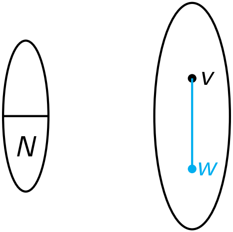

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K_s-saturation in $G(n,p)$

Theorem (K–Sudakov)

Let $0 < p < 1$ be some constant probability and $s \geq 3$ be an integer. Then the minimum number of edges in a K_s-saturated subgraph of $G(n,p)$ is, whp,

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- Does not depend on s.

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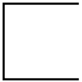

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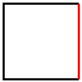

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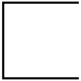

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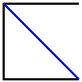

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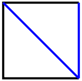

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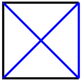

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Other proofs by Kalai, Alon, Yu, etc.

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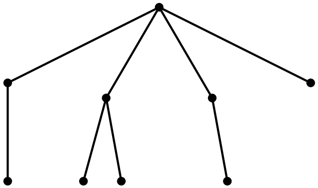

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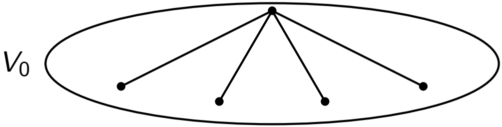

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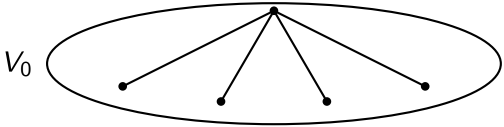

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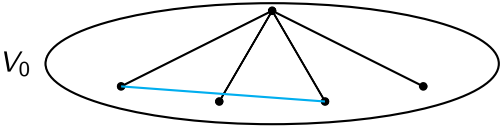

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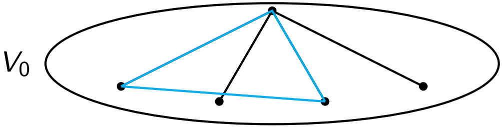

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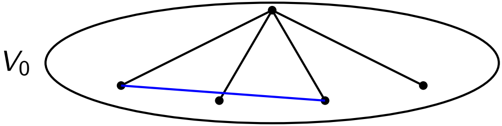

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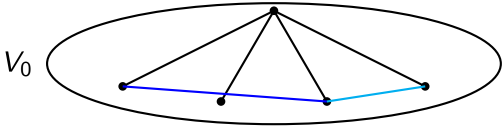

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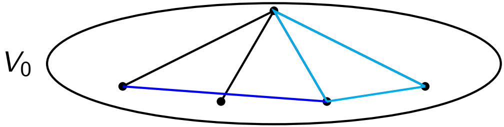

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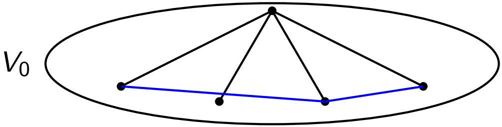

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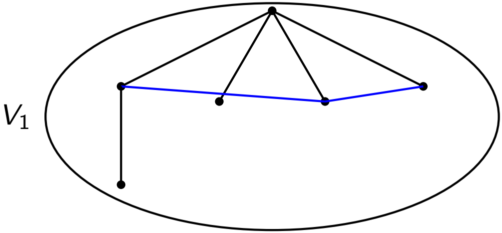

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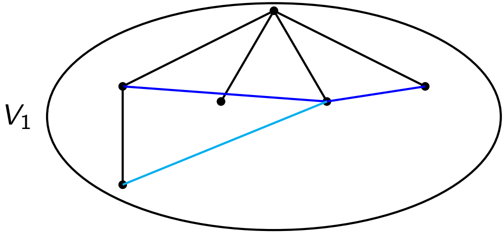

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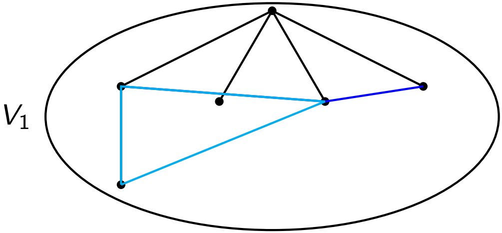

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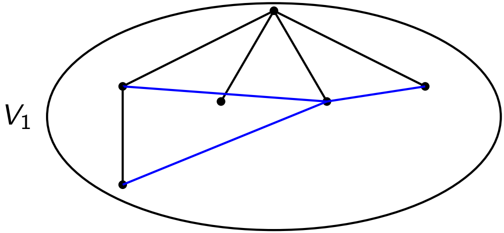

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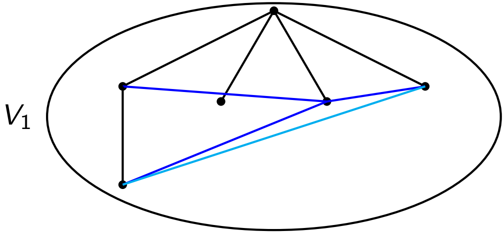

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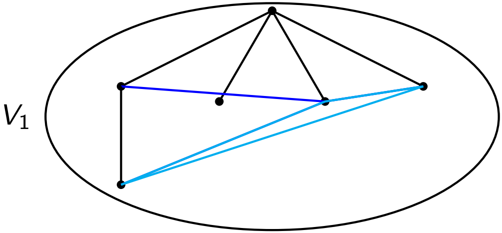

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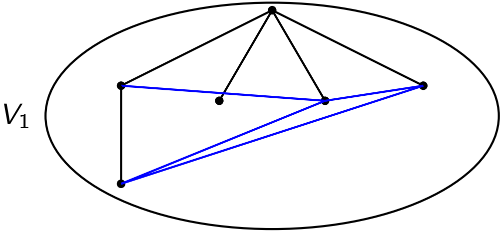

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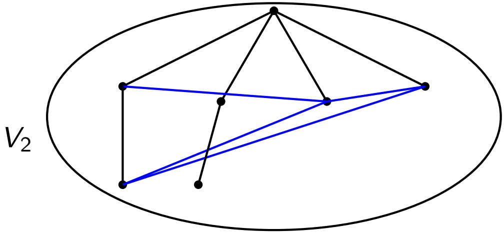

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- This problem might have the same threshold

Thank you!